APPLIED MATHEMATICS SERIES

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FINITE DEFORMATION

OF AN

ELASTIC SOLID

APPLIED MATHEMATICS SERIES

THE APPLIED MATHEMATICS SERIES is devoted to books dealing with mathematical theories underlying physical and biological sciences, and with advanced mathematical techniques needed for solving problems of these sciences

FINITE DEFORMATION OF AN ELASTIC SOLID

FRANCIS D. MURNAGHAN
INSTITUTO TECNÓLOGICO DE AERONÁUTICA
BRAZIL

NEW YORK · JOHN WILEY & SONS, INC.
LONDON · CHAPMAN & HALL, LIMITED
1951

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This book treats, in a self-contained manner, the problem of the deformation of an elastic solid without making the assumption of the classical infinitesimal, or linear, theory of elasticity that the deformation is so small that the squares of the strain components are negligible. Matrices are used methodically, and an account of those properties of matrices which are used in the book is given in the first chapter. The importance of non-isotropic media is emphasized, and the fact that applied stress can make non-isotropic a medium which is isotropic when free from stress is pointed out. The relations between the third-order, as well as the second-order, elastic constants for various types of crystalline media are derived. The agreement between the predictions of theory and Bridgman's experiments on the compressibility of media under hydrostatic pressures up to 10^5 atmospheres is shown. The final chapter applies the theory to the problem of the deformation of spherical shells and circular tubes under extreme external and internal pressures.

We have, so far, very little experimental knowledge of the third-order elastic constants or of the effect of applied stress upon the second-order elastic constants. If the mathematical treatment given here serves to stimulate the procurement of experimental knowledge of these phenomena we shall have attained our aim.

Part of the material of the book was given in lectures at the Dublin Institute for Advanced Studies (summer, 1948), the Carnegie Institute of Technology (fall, 1948), and the Escola Tecnica do Exercito, Rio de Janeiro (1949). We have thought it well to let the book maintain the elementary form of treatment adopted in these lectures and have not attempted to provide references to other treatments of the topics discussed in the book. Owing, doubtless, to technological demands, these topics are currently much discussed, and the interested reader may find many references in the latest issues of such abstract journals as *Mathematical Reviews and Science Abstracts.*

The inscription that closed the preface to my previous book in the Wiley Applied Mathematics Series, *Introduction to Applied Mathematics,* may well be repeated here:

To the Glory of God, Honor of Ireland,
and
Solidarity of the Americas.

Rio de Janeiro
August 1951

F. D. Murnaghan

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1

VECTORS AND MATRICES

The theory of finite deformations of an elastic solid is most easily presented and understood by the use of matrices. Very little beyond the elements of the theory of matrices is needed, but since many engineers who may be interested in the theory of finite deformations may not be familiar with matrices this introductory chapter gives a self-contained and elementary account of that part of the theory of matrices which is necessary for our purpose.

1. The matrix concept

The concept of a matrix arises naturally and inevitably when we consider several differentiable functions of several variables. In elementary calculus we learn that, if $u = u(x)$ is a differentiable function of a single independent variable x, then the differential du of u is obtained by multiplying the differential dx of x by the derivative u_x of u with respect to x:

$$du = u_x\, dx.$$

Thus the derivative u_x of u with respect to x may be regarded as a *magnification factor* that converts, by multiplication, the differential dx of x into the differential du of u. When we pass to the consideration of *two* differentiable functions (u, v) of *two* independent variables (x, y) we have two differentials (du, dv) that are connected with the two differentials (dx, dy) of the independent variables by the formulas

$$du = u_x\, dx + u_y\, dy,$$

$$dv = v_x\, dx + v_y\, dy.$$

Here u_x and u_y are the partial derivatives of $u = u(x, y)$ with respect to x and y, respectively, and v_x and v_y are the partial derivatives of $v = v(x, y)$ with respect to x and y, respectively. The collection of four partial derivatives (u_x, u_y, v_x, v_y) now plays the role previously

played by the one derivative u_x, and this *collection* of four derivatives may be considered as a *magnification factor* that converts, by multiplication, the *pair* of differentials (dx, dy) into the *pair* of differentials (du, dv). We regard the pair of differentials (dx, dy) as a single entity, the differential *vector* of the independent variables, and, similarly, we regard the pair of differentials (du, dv) as a single entity, the differential vector of the dependent variables. We denote these differential vectors by bold-face type, thus $\mathbf{dx}$ is the differential vector of the independent variables and (dx, dy) are the *coordinates* of $\mathbf{dx}$. In symbols

$$\mathbf{dx} = \iota(dx, dy)$$

(the translation of this stenographic notation being $\mathbf{dx}$ is the vector whose coordinates are dx and dy). Similarly,

$$\mathbf{du} = \iota(du, dv)$$

(translate this stenographic symbol). It is convenient to write the coordinates of $\mathbf{dx}$ in a column, i.e. one above the other, thus

$$\begin{pmatrix} dx \\ dy \end{pmatrix}$$

We refer to this symbol as a *2 × 1 matrix*, i.e., a matrix of two rows and one column. The first coordinate dx of the vector $\mathbf{dx}$ is the *element* in the first row of the matrix $\begin{pmatrix} dx \\ dy \end{pmatrix}$, and the second coordinate dy of the vector $\mathbf{dx}$ is the element in the second row of the matrix $\begin{pmatrix} dx \\ dy \end{pmatrix}$. Similarly, we write the coordinates of $\mathbf{du}$ as the elements of a 2×1 matrix $\begin{pmatrix} du \\ dv \end{pmatrix}$, and then the collection of four derivatives (u_x, u_y, v_x, v_y) may be regarded as constituting a magnification factor that converts, by multiplication, the 2×1 matrix $\begin{pmatrix} dx \\ dy \end{pmatrix}$ into the 2×1 matrix $\begin{pmatrix} du \\ dv \end{pmatrix}$. We write the four derivatives as the elements of a 2×2 matrix (i.e., a matrix of two rows and two columns), and we denote this matrix by the symbol $\dfrac{(u, v)}{(x, y)}$.

$$\frac{(u, v)}{(x, y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Note that the upper letters of the symbol $\dfrac{(u,\ v)}{(x,\ y)}$ tell the rows, whereas the lower letters tell the columns, of the matrix. Thus the elements of the first row are the derivatives of u with respect to x and y, and the elements of the second row are the derivatives of v with respect to x and y. On the other hand, the elements of the first column are the derivatives of u and v with respect to x, and the elements of the second column are the derivatives of u and v with respect to y. This 2×2 matrix $\dfrac{(u,\ v)}{(x,\ y)}$ is known as the *Jacobian* matrix (after K. G. J. Jacobi [1801–1851], a German mathematician) of the pair of functions (u, v) with respect to the pair of independent variables (x, y); we denote it by the symbol J. We now introduce a shorthand notation in which we denote any one-column matrix by the symbol for the element in its first row; thus we denote the 2×1 matrix $\begin{pmatrix} dx \\ dy \end{pmatrix}$ simply by dx and the 2×1 matrix $\begin{pmatrix} du \\ dv \end{pmatrix}$ simply by du. In this notation the pair of equations

$$du = u_x\, dx + u_y\, dy,$$

$$dv = v_x\, dx + v_y\, dy$$

appears in the abbreviated form

$$du = J\, dx.$$

The 2×2 matrix J has two 2×1 matrices, or *column vectors* as we shall term them, namely, $\begin{pmatrix} u_x \\ v_x \end{pmatrix}$ and $\begin{pmatrix} u_y \\ v_y \end{pmatrix}$, and two 1×2 matrices, or *row vectors* as we shall call them, namely, (u_x, u_y) and (v_x, v_y). We term the expression $u_x\, dx + u_y\, dy$ (which furnishes the first element of the 2×1 matrix du) the *product* of the 2×1 matrix dx by the first row vector of J; similarly, the expression $v_x\, dx + v_y\, dy$ (which furnishes the second element of the 2×1 matrix du) is the product of the 2×1 matrix dx by the second row vector of J. In this terminology, then, the product $J\, dx$ of the 2×1 matrix dx by the 2×2 Jacobian matrix J is a 2×1 matrix du whose elements are obtained by methodically multiplying the 2×1 matrix dx by the row vectors of J, one after the other.

Example. The formulas connecting plane polar coordinates (ρ, ϕ) and rectangular Cartesian coordinates (x, y) are $x = \rho \cos \phi, y = \rho \sin \phi$. The Jacobian matrix $\dfrac{(x,\ y)}{(\rho,\ \phi)}$ of (x, y) with respect to (ρ, ϕ) is

$$J = \begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix}$$

and the product of the 2×1 matrix $d\rho = \begin{pmatrix} d\rho \\ d\phi \end{pmatrix}$ by J is

$$dx = \begin{pmatrix} \cos\phi\, d\rho - \rho\sin\phi\, d\phi \\ \sin\phi\, d\rho + \rho\cos\phi\, d\phi \end{pmatrix}$$

The first element of this matrix is the differential of x and the second element is the differential of y

Remark You should note the analogy between the relation $du = J\,dx$ and the basic relation $dy = y_x\,dx$ of elementary differential calculus Realize clearly that it is the 2×2 Jacobian matrix J that plays for two differentiable functions of two independent variables the role played by the *derivative* of one differentiable function of one independent variable The four elements of J have individually no fundamental significance It is the 2×2 Jacobian *matrix* of which they are the elements that is the important thing

There is no difficulty in extending the theory of the preceding paragraphs to three differentiable functions of three independent variables or generally to n differentiable functions of n independent variables $n = 3\ 4\ 5$ Thus for three independent variables we denote by dx the 3×1 matrix $\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ and by du the 3×1 matrix $\begin{pmatrix} du \\ dv \\ dw \end{pmatrix}$

The Jacobian matrix $J = \dfrac{(u\ \ v\ \ w)}{(x\ \ y\ \ z)}$ is the matrix

$$J = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

of which for example the 1×3 matrix $(u_x\ \ u_y\ \ u_z)$ is the first row vector J may be regarded as a magnification factor that converts by multiplication the 3×1 matrix dx into the 3×1 matrix du

$$du = J\,dx$$

In this kind of multiplication of dx by J each element of the 3×1 matrix du is the product of the 3×1 matrix dx by the corresponding row vector of J for example the element dv in the second row of the 3×1 matrix du is given by the formula

$$dv = v_x\,dx + v_y\,dy + v_z\,dz$$

Example. The formulas connecting space polar coordinates (r, θ, ϕ) and rectangular Cartesian coordinates (x, y, z) are

$$x = r \sin \theta \cos \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \theta.$$

The Jacobian matrix $J = \dfrac{(x, y, z)}{(r, \theta, \phi)}$ of (x, y, z) with respect to (r, θ, ϕ) is

$$J = \begin{pmatrix} \sin \theta \cos \phi & r \cos \theta \cos \rho & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix}.$$

The elements of the 3×1 matrix dx are furnished by the formula $dx = J\,dr$; thus

$$dx = \sin \theta \cos \phi\, dr + r \cos \theta \cos \phi\, d\theta - r \sin \theta \sin \phi\, d\phi,$$

$$dy = \sin \theta \sin \phi\, dr + r \cos \theta \sin \phi\, d\theta + r \sin \theta \cos \phi\, d\phi,$$

$$dz = \cos \theta\, dr - r \sin \theta\, d\theta.$$

It should be clear from the preceding discussion what is meant by an $m \times n$ matrix A where m and n are any two positive integers. This is a collection of mn numbers arranged in m rows and n columns. We denote the *element* in the rth row and sth column by the symbol a_s^r. Note carefully that the upper label tells the row and the lower label tells the column. In the applications of matrix theory which we shall have to make neither m nor n will surpass 3, and we can use the readily visualized concepts of one-dimensional vectors (i.e., vectors on a line) or of two-dimensional vectors (i.e., plane vectors) or of three-dimensional vectors (i.e., space vectors). For example, if $m = 2$, $n = 3$, so that A has two rows and three columns, A possesses three plane column vectors of which the second, for instance, is $\begin{pmatrix} a_2^1 \\ a_2^2 \end{pmatrix}$, and it also possesses two space row vectors of which the first, for instance, is (a_1^1, a_2^1, a_3^1). If B is an $n \times p$ matrix (so that B possesses as many rows as A possesses columns) it is possible to multiply any column vector of B by any row vector of A since the column vectors of B and the row vectors of A are all of the same dimension, namely, n. We write these various products of column vectors of B by row vectors of A as an $m \times p$ matrix C, the element in the jth row and kth column of C being the product of the kth column vector of B by the jth row

vector of A, $j = 1, \quad m$, $k = 1, \quad , p$, and we term C the *product* of the $n \times p$ matrix B by the $m \times n$ matrix A We indicate this product operation as follows

$$C = AB$$

EXERCISES

1. Show that if A is polite in multiplication, i e , if AB is the same as B for every B, A must be *square* (i e , have the same number of columns as rows) with all its diagonal elements 1 and all its other elements zero

Note The diagonal elements of a square matrix A are the elements $a_1^1, a_2^2,$ whose row and column numbers are equal

Hint If A is an $m \times n$ matrix and B an $n \times p$ matrix, AB is an $m \times p$ matrix For AB to be the same as B we must have $m = n$ so that A is square

Remark The matrices that are polite in multiplication are known as *unit* matrices Terming the common number of rows and columns of a square matrix its *dimension,* we denote the unit matrix of dimension n by the symbol E_n Thus

$$E_1 = (1), \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Form the following products

$$(a)\ \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix}, \qquad (b)\ \begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix},$$

$$(c)\ \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} (2), \qquad (d)\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

Answer $(a)\ \begin{pmatrix} 14 & 13 \\ -2 & -5 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 8 & 13 \\ 4 & 1 \end{pmatrix}, \quad (c)\ \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}, \quad (d)\ \begin{pmatrix} 6 \\ 2 \\ 14 \end{pmatrix}$

Note The difference between the answers to (a) and (b) shows that when both products AB and BA can be formed (show that each product is a square matrix when this happens) and when they have the same dimension (show that A and B are each square when this happens) the products are not necessarily the same *Multiplication of square matrices of the same dimension is not, in general, commutative* The results of (c) and (d) show that if we denote by cE_n the $n \times n$ matrix whose diagonal elements are all $= c$ and whose non-diagonal elements are zero then if A is any $p \times q$ matrix each of the products $(cE_p)A$ and $A(cE_q)$ is obtained by multiplying each and every element of A by c We denote each of these products simply by cA

3 Form the two products

$$(a)\ (-4, 1, 2) \begin{pmatrix} 6 \\ -5 \\ 2 \end{pmatrix} \quad \text{and} \quad (b)\ \begin{pmatrix} 6 \\ -5 \\ 2 \end{pmatrix} (-4, 1, 2)$$

Answer $(a)\ (-25), \quad (b)\ \begin{pmatrix} -24 & 6 & 12 \\ 20 & -5 & -10 \\ -8 & 2 & 4 \end{pmatrix}$

4. Show that, if D is a diagonal matrix, DA is obtained by multiplying each row vector of A by the corresponding diagonal element of D whereas AD is obtained by multiplying each column vector of A by the corresponding diagonal element of D. *Note.* A *diagonal* matrix D is a square matrix of which every non-diagonal element is zero; thus $d_j^k = 0$ if $k \neq j$.

5. Show that if D_1 and D_2 are any two diagonal matrices of the same dimension then $D_2D_1 = D_1D_2$; in words, *multiplication of diagonal matrices is commutative.*

6. Show that if A is any $m \times n$ matrix and B is any $n \times m$ matrix, so that both AB and BA are square matrices, then the sum of the diagonal elements of AB is the same as the sum of the diagonal elements of BA. *Hint.* The sum of the diagonal elements of either AB or BA is the sum of all products obtained by multiplying any element of A by the associated element of B where we understand that the element of B which is associated with $a_p{}^q$ is $b_q{}^p$ (note the interchange of row and column; for example, the element of B which is associated with the element in the second row and third column of A lies in the third row and second column of B). *Note.* We term the sum of the diagonal elements of any square matrix the *trace* of the matrix, and we denote the trace of A by $Tr\ A$. The result of the present exercise appears then as follows:

$$Tr\ AB = Tr\ BA$$

(although AB and BA are in general different and, indeed, in general of different dimensions).

7. Verify the result of Exercise 6 for the two matrices of Exercise 3 and for the two matrices $A = (a_1,\ a_2,\ a_3)$ and $B = \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}$. What are the dimensions of AB and BA? Note that AB is the product of the column vector B by the row vector A.

8. Show that matrix multiplication is associative; i.e., if A is any $m \times n$ matrix, B any $n \times p$ matrix, and C any $p \times q$ matrix then both the products $(AB)C$ and $A(BC)$ may be formed and they turn out to be the same. *Hint.* The element in the rth row and sth column of either product is the double sum $\sum_{i=1}^{n} \sum_{j=1}^{p} a_i{}^r\, b_j{}^i\, c_s{}^j$. *Note.* We denote the common product simply by ABC.

9. Show that matrix multiplication is distributive with respect to addition, i.e., that $(A + B)C = AC + BC$. *Note.* Addition of matrices is defined only for matrices of the same *type*, i.e., of the same number of rows and columns, and the rule of addition may be phrased as follows: *Respect the row and column position.* Thus if A and B are each $m \times n$ matrices the element in the pth row and qth column of $A + B$ is (by definition)

$$a_q{}^p + b_q{}^p.$$

10. Show that addition of matrices is a commutative operation (what does this mean?).

11. Show that $A + A = 2A$.

12. Show that $Tr\ (A + B) = Tr\ A + Tr\ B$.

13. Show that the *transpose* of $A + B$ is the sum of the transposes of A and B. *Note.* If A is any $m \times n$ matrix we term the matrix obtained by interchanging the rows and columns of A the *transpose* of A and we denote the transpose of A by an

attached star A^* Thus the present exercise may be written as follows $(A + B)^* = A^* + B^*$ The row vectors of A^* are the column vectors of A, and the column vectors of A^* are the row vectors of A

14 State in words and prove that $(A^*)^* = A$

15 Form the cofactor matrix of each of the matrices $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$, $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *Note* If A is any square matrix of dimension n we understand by the cofactor matrix of A (denoted by co A) the matrix obtained by replacing each element of A by its cofactor the cofactor of $a_p{}^q$ being the product of the determinant of the $(n-1)$-dimensional matrix obtained by erasing the qth row and pth column of A by $(-1)^{p+q}$

Answer $\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}$ $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

16 If A is a square matrix of dimension n show that $A(\text{co } A)^* = (\det A)\, E_n$ where det A denotes the determinant of A *Note* We shall not stop for a definition and discussion of the elementary concept of the determinant of a square matrix Note, however that determinants may be *defined*, by induction from $n-1$ to n by means of the result of this exercise, it being understood that the determinant of a one-dimensional matrix is the (one and only) element of the matrix Thus to define det A where A is the two-dimensional matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ we have, first, co $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (see Exercise 15) Hence $(\text{co } A)^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and so $A(\text{co } A)^* = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad-bc)E_2$ Hence $\det A = ad - bc$

17. Calculate co A and det A for $A = \begin{pmatrix} 2 & 4 & 5 \\ 0 & -1 & -1 \\ -3 & -2 & 0 \end{pmatrix}$

Answer co $A = \begin{pmatrix} 2 & 3 & -3 \\ 10 & 15 & -16 \\ 1 & 2 & -2 \end{pmatrix}$ $\det A = 1$

18 Show that if A is a square matrix of dimension n whose determinant is not zero the matrix $B = (\det A)^{-1}\,(\text{co } A)^*$ has the property that $AB = BA = E_n$ *Note* We term the matrix B defined in this way the *reciprocal* of A and we denote it by the symbol A^{-1} A square matrix whose determinant is not zero is termed *non singular*

19 Show that if $AB = E_n$, where A is a square matrix of dimension n, then A is non-singular and $B = A^{-1}$ *Hint* A is non-singular since $\det(AB) = \det E_n = 1$ and $\det(AB) = \det A \det B$ On multiplying the relation $AB = E_n$ through on the left by A^{-1} and using the associative property of matrix multiplication we obtain $B = A^{-1}$

20 Show that if $BA = E_n$, where A is a square matrix of dimension n, then A is non-singular and $B = A^{-1}$ *Note* The results of Exercises 19 and 20 may be briefly expressed as follows Only non-singular matrices A can satisfy either of the relations $AB = E_n$, $BA = E_n$, and when A is non-singular the associated matrix B is unambiguously determinate, being A^{-1}

21. Show that if A and B are non-singular matrices of dimension n so also is AB and determine $(AB)^{-1}$. *Hint.* $(B^{-1}A^{-1})(AB) = E_n$.

Answer. $(AB)^{-1} = B^{-1}A^{-1}$.

Note. The important result of this exercise may be phrased as follows: *The reciprocal of the product of two non-singular matrices of dimension n is the product of their reciprocals in the reverse order.*

22. Generalize the result of Exercise 21 to the product of any number of non-singular matrices of dimension n.

23. Show that if A is any $m \times n$ matrix and B any $n \times p$ matrix, so that the product AB can be formed, then the product B^*A^* can be formed and $B^*A^* = (AB)^*$.

24. Generalize the result of Exercise 23 to the product of any number of matrices.

Note. The important result of this exercise may be phrased as follows: *The transpose of the product of any number of matrices is the product of their transposes in the reverse order.*

25. Show that if A is any square matrix then $Tr\,A^* = Tr\,A$. *Hint.* The diagonal elements of A^* are the same as the diagonal elements of A.

2. The matrix element of arc and the element of area in the plane

Let the rectangular Cartesian coordinates (x, y) of a variable point P in a plane be functions of a single independent variable, or parameter, α. Then P traces, as α varies, a curve (which reduces to a point if x and y are each constant functions of α). We term the 2×1 matrix

$$dx = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_\alpha\, d\alpha \\ y_\alpha\, d\alpha \end{pmatrix}$$

the *matrix element of arc* of the curve that is traced by P. The squared magnitude of the vector $v(dx, dy)$ is furnished by the formula

$$(ds)^2 = (dx)^2 + (dy)^2 = (dx)^*\, dx$$

where the 1×2 matrix $(dx)^* = (dx, dy)$ is the transpose of the 2×1 matrix dx. We term ds, i.e., the *positive* square root of $(dx)^*\, dx$, the *scalar element of arc* of the curve traced by P.

Let, now, the rectangular Cartesian coordinates (x, y) of P be functions of two independent variables, or parameters, α and β. We have now two 2×1 matrix elements of arc $d_\alpha x$ and $d_\beta x$ where

$$d_\alpha x = \begin{pmatrix} x_\alpha\, d\alpha \\ y_\alpha\, d\alpha \end{pmatrix}, \qquad d_\beta x = \begin{pmatrix} x_\beta\, d\beta \\ y_\beta\, d\beta \end{pmatrix}$$

and we set up the 2×2 matrix whose column vectors are $d_\alpha x$ and $d_\beta x$:

$$\begin{pmatrix} x_\alpha\, d\alpha & x_\beta\, d\beta \\ y_\alpha\, d\alpha & y_\beta\, d\beta \end{pmatrix}.$$

The absolute value of the determinant of this matrix is the *element of area* in the plane, it being understood that this determinant is not zero, i e, that $\det \dfrac{(x\ y)}{(\alpha,\ \beta)} \neq 0$ Since an interchange of the parameters α and β changes the sign of $\det \dfrac{(x,\ y)}{(\alpha,\ \beta)}$ we may choose these parameters in such an order that $\det \dfrac{(x\ y)}{(\alpha\ \beta)} > 0$ and we suppose this done Denoting the element of area in the plane by dS, we have, then,

$$dS = \det \frac{(x,\ y)}{(\alpha,\ \beta)}\, d\alpha\, d\beta$$

It frequently happens that x and y are not given directly as functions of the independent variables α and β but rather as functions of two other variables, which are functions of α and β For example, in the theory of plane deformations a point P_a whose coordinates relative to any convenient rectangular Cartesian reference frame are $(a,\ b)$ is deformed into a point P_x whose coordinates relative to any convenient rectangular Cartesian reference frame (not necessarily the same as before) are $(x,\ y)$ We have now two elements of area, an *initial* element of area dS_a defined by

$$dS_a = \det \frac{(a,\ b)}{(\alpha,\ \beta)}\, d\alpha\, d\beta$$

and a *final* element of area dS_x defined by

$$dS_x = \det \frac{(x,\ y)}{(\alpha\ \beta)}\, d\alpha\, d\beta$$

It follows immediately from the rule of multiplication of determinants (in accordance with which the determinant of the product of two square matrices is the product of the determinants of these matrices) that

$$dS_x = \det \frac{(x,\ y)}{(a,\ b)}\, dS_a$$

$\left(\text{Prove this}\quad \textit{Hint}\quad \dfrac{(x,\ y)}{(\alpha,\ \beta)} = \dfrac{(x,\ y)}{(a,\ b)}\, \dfrac{(a,\ b)}{(\alpha\ \beta)}\right)$ The easiest and quickest way of evaluating dS_x is to proceed according to the following formal procedure Write down $dx = x_a\, da + x_b\, db$, $dy = y_a\, da + y_b\, db$

and multiply dy by dx where it is understood that the differentials da and db obey the following *alternating* rules of multiplication:

$$da\,da = 0; \qquad db\,da = -da\,db; \qquad db\,db = 0.$$

We find that $dx\,dy = (x_a y_b - x_b y_a) da\,db$, and on setting $da\,db = dS_a$ we see that dS_x is $dx\,dy$.

Example 1. Consider the plane *shear* defined by the formulas

$$x = a + kb,$$
$$y = \qquad b$$

where k is a constant. We have $dx = da + k\,db$, $dy = db$, and so

$$dS_x = dx\,dy = da\,db = dS_a.$$

Thus the final element of area dS_x is the same as the initial element of area dS_a.

Example 2. Let the independent variables (α, β) be plane polar coordinates (r, θ). From the equations $x = r\cos\theta$, $y = r\sin\theta$ we read off $dx = \cos\theta\,dr - r\sin\theta\,d\theta$, $dy = \sin\theta\,dr + r\cos\theta\,d\theta$, and on evaluating $dx\,dy$ (where dr and $d\theta$ obey the alternating rules of multiplication, $dr\,dr = 0$, $d\theta\,dr = -dr\,d\theta$, $d\theta\,d\theta = 0$) we obtain $dx\,dy = r(\cos^2\theta + \sin^2\theta)\;dr\,d\theta = r\,dr\,d\theta$. In words, the element of area in plane polar coordinates is $r\,dr\,d\theta$.

EXERCISES

1. Show that, in the plane shear defined by $x = a + kb$, $y = b$, vectors parallel to the a-axis are unchanged whereas vectors parallel to the b-axis are turned through the angle arc tan k and are increased in magnitude by the factor $(1 + k^2)^{1/2}$. *Hint.* $dx = da + k\,db$, $dy = db$. When $db = 0$, $v(dx, dy) = v(da, db)$, and, when $da = 0$, $v(dx, dy) = v(k\,db, db)$.

2. Determine the effect of the plane shear of Exercise 1 on a vector that is inclined at an angle of $\pi/4$ to the a-axis.

3. Show that in the *plane torsion* defined by

$$\left.\begin{aligned} x &= r\cos(\theta + k) = a\cos k - b\sin k \\ y &= r\sin(\theta + k) = a\sin k + b\cos k \end{aligned}\right\}, \qquad k \text{ constant},$$

the final element of area dS_x is the same as the initial element of area dS_a. Show also that the magnitude of any vector in the plane is unaffected by the plane torsion.

3. Matrix elements of arc and of surface area and the element of volume in space

Let the rectangular Cartesian coordinates (x, y, z) of a variable point P in space be functions of a single independent variable, or parameter, α.

Then P traces, as α varies, a curve (which reduces to a point if x, y, and z are each constant functions of α) We term the 3×1 matrix

$$dx = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} x_\alpha \, d\alpha \\ y_\alpha \, d\alpha \\ z_\alpha \, d\alpha \end{pmatrix}$$

the matrix element of arc of the curve that is traced by P The squared magnitude of the vector $v(dx, dy, dz)$ is furnished by the formula

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dx)^* \, dx$$

where the 1×3 matrix $(dx)^* = (dx, dy, dz)$ is the transpose of the 3×1 matrix dx We term ds (i e , the positive square root of $(dx)^* \, dx$) the *scalar element of arc* of the curve traced by P

Let, now, the rectangular Cartesian coordinates (x, y, z) of P be functions of two independent variables, or parameters, α and β We have now two 3×1 matrix elements of arc $d_\alpha x$ and $d_\beta x$ where

$$d_\alpha x = \begin{pmatrix} x_\alpha \, d\alpha \\ y_\alpha \, d\alpha \\ z_\alpha \, d\alpha \end{pmatrix}, \qquad d_\beta x = \begin{pmatrix} x_\beta \, d\beta \\ y_\beta \, d\beta \\ z_\beta \, d\beta \end{pmatrix},$$

and we set up the 3×2 matrix whose column vectors are $d_\alpha x$ and $d_\beta x$

$$\begin{pmatrix} x_\alpha \, d\alpha & x_\beta \, d\beta \\ y_\alpha \, d\alpha & y_\beta \, d\beta \\ z_\alpha \, d\alpha & z_\beta \, d\beta \end{pmatrix}$$

If we regard this 3×2 matrix as obtained by erasing the first column of a 3×3 matrix the first column of the cofactor matrix of this 3×3 matrix is $\begin{pmatrix} dS^x \\ dS^y \\ dS^z \end{pmatrix}$ where $dS^x = \det \dfrac{(y, z)}{(\alpha, \beta)} \, d\alpha \, d\beta$, $dS^y = \det \dfrac{(z, x)}{(\alpha, \beta)} \, d\alpha \, d\beta$, $dS^z = \det \dfrac{(x, y)}{(\alpha, \beta)} \, d\alpha \, d\beta$ We term this 3×1 matrix the *matrix element of area* of the surface traced out by P The vector $v(dS^x, dS^y, dS^z)$ is the vector, or cross, product $\mathbf{d}_\alpha \mathbf{x} \times \mathbf{d}_\beta \mathbf{x}$, and so $v(dS^x, dS^y, dS^z)$ has the direction of one of the two normal directions at P to the surface traced out by P (it being understood that $\mathbf{d}_\alpha \mathbf{x} \times \mathbf{d}_\beta \mathbf{x}$ is not the zero vector) In other words, the column vector of the matrix element of area is normal to the surface If dS^x denotes the matrix element of area the *scalar element of area* dS_x is the positive square root of

$(dS^x)^* \, dS^x$ where $(dS^x)^*$, the transpose of dS^x, is the 1×3 matrix (dS^x, dS^y, dS^z).

It frequently happens that x, y, and z are not given directly as functions of the independent variables α and β but are given rather as functions of three other variables, which are functions of α and β. For example, in the theory of space deformations a point P_a whose coordinates relative to any convenient rectangular Cartesian reference frame are (a, b, c) is deformed into a point P_x whose coordinates relative to any convenient rectangular Cartesian reference frame (not necessarily the same as before) are (x, y, z). We now have two matrix elements of area: an *initial* matrix element of area,

$$dS^a = \begin{pmatrix} dS^a \\ dS^b \\ dS^c \end{pmatrix} = \begin{pmatrix} \det \dfrac{(b,\ c)}{(\alpha,\ \beta)} \, d\alpha \, d\beta \\ \det \dfrac{(c,\ a)}{(\alpha,\ \beta)} \, d\alpha \, d\beta \\ \det \dfrac{(a,\ b)}{(\alpha,\ \beta)} \, d\alpha \, d\beta \end{pmatrix},$$

and a *final* matrix element of area,

$$dS^x = \begin{pmatrix} dS^x \\ dS^y \\ dS^z \end{pmatrix} = \begin{pmatrix} \det \dfrac{(y,\ z)}{(\alpha,\ \beta)} \, d\alpha \, d\beta \\ \det \dfrac{(z,\ x)}{(\alpha,\ \beta)} \, d\alpha \, d\beta \\ \det \dfrac{(x,\ y)}{(\alpha,\ \beta)} \, d\alpha \, d\beta \end{pmatrix}.$$

Adopting the alternating rule of multiplication of differentials (in accordance with which $d\alpha \, d\alpha = 0$, $d\beta \, d\alpha = -d\alpha \, d\beta$, $d\beta \, d\beta = 0$), we may write dS^a in the more compact form

$$dS^a = \begin{pmatrix} db \, dc \\ dc \, da \\ da \, db \end{pmatrix},$$

and we may write dS^x in the more compact form

$$dS^x = \begin{pmatrix} dy \, dz \\ dz \, dx \\ dx \, dy \end{pmatrix}.$$

Since $y_\alpha = y_a a_\alpha + y_b b_\alpha + y_c c_\alpha$, etc, we have

$$\det\frac{(y,\, z)}{(\alpha,\, \beta)} = \det\frac{(y,\, z)}{(b,\, c)}\det\frac{(b,\, c)}{(\alpha,\, \beta)} + \det\frac{(y,\, z)}{(c,\, a)}\det\frac{(c,\, a)}{(\alpha,\, \beta)} + \det\frac{(y,\, z)}{(a,\, b)}\det\frac{(a,\, b)}{(\alpha,\, \beta)}, \text{ etc},$$

(prove this), and so

$$dS^x = \det\frac{(y,\, z)}{(b,\, c)}\, dS^a + \det\frac{(y,\, z)}{(c,\, a)}\, dS^b + \det\frac{(y,\, z)}{(a,\, b)}\, dS^c$$

This and the two similar equations, which furnish dS^y and dS^z (write down these equations), tell us that the 3×1 matrix dS^x is the product of the 3×1 matrix dS^a by the 3×3 matrix $\mathrm{co}\, J$ where J is the Jacobian matrix $\dfrac{(x,\, y,\, z)}{(a\;\; b,\, c)}$

$$dS^x = (\mathrm{co}\, J)\, dS^a$$

Taking it for granted that J is non singular (i e, that $\det J \neq 0$), we may write this relation in the equivalent form

$$dS^x = (\det J)(J^*)^{-1}\, dS^a$$

Note that $\det J = \det J^*$ and that $(J^*)^{-1} = \left(\dfrac{1}{\det J^*}\right)\mathrm{co}\, J$ It is clear from the relations $dS^x = \det\dfrac{(y,\, z)}{(b,\, c)}\, dS^a + \det\dfrac{(y,\, z)}{(c,\, a)}\, dS^b + \det\dfrac{(y,\, z)}{(a,\, b)}\, dS^c$, etc, that dS^x may be calculated by multiplying out $dy\, dz = (y_a\, da + y_b\, db + y_c\, dc)(z_a\, da + z_b\, db + z_c\, dc)$, subjecting the differentials da, db, dc to the following alternating rules of multiplication $da\, da = 0 = db\, db = dc\, dc$, $dc\, db = -db\, dc$, $da\, dc = -dc\, da$, $db\, da = -da\, db$, and finally setting $db\, dc = dS^a$, $dc\, da = dS^b$, $da\, db = dS^c$

Example 1. Determine the matrix element of area of the sphere $x = r \sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, r constant We have $dy = r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$, $dz = -r\sin\theta\, d\theta$, and so $dS^x = dy\, dz = -r^2\sin^2\theta\cos\phi\, d\phi\, d\theta = r^2\sin^2\theta\cos\phi\, d\theta\, d\phi$ Similarly, (show this) $dS^y = dz\, dx = r^2\sin^2\theta\sin\phi\, d\theta\, d\phi$ and $dS^z = r^2\sin\theta\cos\theta\, d\theta\, d\phi$, and thus the 3×1 matrix element of area is

$$dS^x = r^2\sin\theta\begin{pmatrix}\sin\theta & \cos\phi\\ \sin\theta & \sin\phi\\ \cos\theta & \end{pmatrix} d\theta\, d\phi$$

Hence (show this) the scalar element of area $dS_x = r^2 \sin\theta\, d\theta\, d\phi$, and the column vector of the matrix element of area has the direction of the outward drawn radius.

Example 2. Determine the matrix elements of area of the surface into which the surface of a rectangular box whose faces are parallel to the various coordinate planes is deformed by the shear $x = a + kc$, $y = b$, $z = c$, it being understood that k is a constant and that the point $P_a:(a, b, c)$ is deformed into the point $P_x:(x, y, z)$.

We have $dx = da + k\,dc$, $dy = db$, $dz = dc$, and so

$$dS^x = \begin{pmatrix} dy\, dz \\ dz\, dx \\ dx\, dy \end{pmatrix} = \begin{pmatrix} db\, dc \\ dc\, da \\ da\, db + k\, dc\, db \end{pmatrix} = \begin{pmatrix} dS^a \\ dS^b \\ dS^c - k\, dS^a \end{pmatrix}.$$

Thus the matrix element of area of a face of the box which is parallel to the a-coordinate plane (for which $dS^b = 0 = dS^c$) is changed by the shear from $\begin{pmatrix} dS^a \\ 0 \\ 0 \end{pmatrix}$ to $dS^a \begin{pmatrix} 1 \\ 0 \\ -k \end{pmatrix}$. The scalar element of area is magnified by the factor $(1 + k^2)^{1/2}$. The matrix element of area of a face of the box which is parallel to the b-coordinate plane (for which $dS^a = 0 = dS^c$) is unchanged by the shear (prove this), and, similarly, the matrix element of area of a face of the box which is parallel to the c-coordinate plane (for which $dS^a = 0 = dS^b$) is unchanged by the shear (prove this).

Let, now, the rectangular Cartesian coordinates (x, y, z) of P be functions of three independent variables, or parameters, α, β, and γ. We have now three 3×1 matrix elements of arc $d_\alpha x$, $d_\beta x$, and $d_\gamma x$ where

$$d_\alpha x = \begin{pmatrix} x_\alpha\, d\alpha \\ y_\alpha\, d\alpha \\ z_\alpha\, d\alpha \end{pmatrix}, \quad d_\beta x = \begin{pmatrix} x_\beta\, d\beta \\ y_\beta\, d\beta \\ z_\beta\, d\beta \end{pmatrix}, \quad d_\gamma x = \begin{pmatrix} x_\gamma\, d\gamma \\ y_\gamma\, d\gamma \\ z_\gamma\, d\gamma \end{pmatrix},$$

and we set up the 3×3 matrix whose column vectors are $d_\alpha x$, $d_\beta x$, and $d_\gamma x$:

$$\begin{pmatrix} x_\alpha\, d\alpha & x_\beta\, d\beta & x_\gamma\, d\gamma \\ y_\alpha\, d\alpha & y_\beta\, d\beta & y_\gamma\, d\gamma \\ z_\alpha\, d\alpha & z_\beta\, d\beta & z_\gamma\, d\gamma \end{pmatrix}.$$

The absolute value of the determinant of this matrix is the *element of volume* in space, it being understood that this determinant is not zero, i.e., that $\det \dfrac{(x, y, z)}{(\alpha, \beta, \gamma)} \neq 0$. We can order the independent variables

so that $\det \dfrac{(x, y, z)}{(\alpha, \beta, \gamma)} > 0$ (show this), thus, on denoting the element of volume by dV_x, we have

$$dV_x = \det \frac{(x, y, z)}{(\alpha, \beta, \gamma)} \, d\alpha \, d\beta \, d\gamma$$

It frequently happens that x, y, and z are not given directly as functions of the independent variables α, β, and γ but rather as functions of three other variables, which are functions of α, β, and γ For example, in the theory of the deformation of a three-dimensional medium, a point P_a whose coordinates relative to any convenient rectangular Cartesian reference frame are (a, b, c) is deformed into a point P_x whose coordinates relative to any convenient rectangular Cartesian reference frame (not necessarily the same as before) are (x, y, z) We have now two elements of volume an *initial* element of volume dV_a, defined by

$$dV_a = \det \frac{(a, b, c)}{(\alpha, \beta, \gamma)} \, d\alpha \, d\beta \, d\gamma,$$

and a *final* element of volume dV_x, defined by

$$dV_x = \det \frac{(x, y, z)}{(\alpha, \beta, \gamma)} \, d\alpha \, d\beta \, d\gamma$$

(it being understood that the independent variables α, β, γ have been so ordered that dV_a and dV_x, defined in this way, are positive) It follows immediately from the rule of multiplication of determinants that

$$dV_x = \det \frac{(x, y, z)}{(a, b, c)} \, dV_a = (\det J) \, dV_a$$

where J is the Jacobian matrix of (x, y, z) with respect to (a, b, c) (prove this) An easy way to evaluate dV_x is to multiply out $dx\,dy\,dz = (x_a\,da + x_b\,db + x_c\,dc)(y_a\,da + y_b\,db + y_c\,dc)(z_a\,da + z_b\,db + z_c\,dc)$ (it being understood that the differentials da, db, dc obey the alternating rules of multiplication $da\,da\,da = 0 = da\,da\,db =$ etc, $da\,dc\,db = -da\,db\,dc$, etc) and to set $da\,db\,dc = dV_a$ If ρ_a and ρ_x denote the initial and final densities of the medium that is being deformed the principle of *conservation of mass* tells us that

$$\rho_x \, dV_x = \rho_a \, dV_a,$$

and so the relation $dV_x - (\det J) \, dV_a$ may be written in the equivalent form

$$\rho_a = \rho_x \det J$$

EXERCISES

1. Show that the final 3×1 matrix element of area dS^x is connected with the initial 3×1 matrix element of area dS^a by the formula

$$dS^x = \frac{\rho_a}{\rho_x} (J^*)^{-1}\, dS^a.$$

2. Show that volumes are preserved under the shear $x = a + kc$, $y = b$, $z = c$. *Hint.* $dx\, dy\, dz = da\, db\, dc$.

3. Show that volumes are preserved under the torsion $x = r \cos (\theta + kc) = a \cos kc - b \sin kc$, $y = r \sin (\theta + kc) = a \sin kc + b \cos kc$, $z = c$, k constant.

4. Show that the element of volume in cylindrical coordinates (r, θ, z), where $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ is $r\, dr\, d\, \theta\, dz$.

5. Show that the element of volume in space polar coordinates (r, θ, ϕ), where $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ is $r^2 \sin \theta\, dr\, d\, \theta\, d\, \phi$.

4. The canonical form of a linear operator

The discussion throughout the present section will concern a square matrix of dimension 3 but the argument is applicable to square matrices of any dimension (repeat the discussion in full detail for a square matrix of dimension 2). We have seen that the Jacobian matrix $J = \dfrac{(u, v, w)}{(x, y, z)}$ of three differentiable functions (u, v, w) of (x, y, z) may be regarded as a magnification factor that converts, by multiplication, the 3×1 matrix dx into the 3×1 matrix du:

$$du = J\, dx.$$

Similarly, any 3×3 matrix M may be regarded as a magnification factor that converts, by multiplication, a 3×1 matrix a into a 3×1 matrix b:

$$b = Ma.$$

Let us now consider the space vectors $\mathfrak{a}$ and $\mathfrak{b}$ whose coordinates relative to any convenient rectangular Cartesian reference frame are the elements of a and b, respectively:

$$\mathfrak{a} = v(a^1, a^2, a^3), \qquad \mathfrak{b} = v(b^1, b^2, b^3).$$

Let the coordinates of $\mathfrak{a}$ and $\mathfrak{b}$ relative to a second rectangular Cartesian reference frame (which may be obtained from the first coordinate reference frame by a rigid rotation) be furnished by the 3×1 matrices a' and b', respectively. (*Note.* A prime is often used to indicate the transpose of a matrix but we have adopted a star to indicate the transpose and this leaves the prime free for the present purpose; a' is *not* the transpose of a; in fact, a' is, like a, a 3×1 matrix, whereas the transpose a^* of a is a 1×3 matrix.) Let the coordinates of unit

vectors along the axes of the new reference frame relative to the original reference frame be furnished by the three 3×1 matrices u_1, u_2, and u_3, and denote by R the 3×3 matrix whose column vectors are u_1, u_2, and u_3 The fact that the three unit vectors along the axes of the new reference frame are not only of unit magnitude but are also mutually perpendicular is conveniently expressed by the matrix equation

$$R^*R = E_3$$

(prove the validity of this equation) The fact that $\det R^* = \det R$ assures us (why?) that $\det R = \pm 1$, and the fact that the second reference frame may be obtained from the first by a rigid rotation restricts us to the relation $\det R = 1$ (why?) (*Hint* $\det R$ is a continuous function of the elements of R, hence it can assume only one or the other of the values ± 1 Since it is originally, when $R = E_3$, $+1$, it must be permanently $+1$) Thus R is a 3×3 matrix whose determinant is $+1$ and whose reciprocal is its transpose We term any such 3×3 matrix a three dimensional *rotation matrix* If $c = \begin{pmatrix} c^1 \\ c^2 \\ c^3 \end{pmatrix}$ is any 3×1 matrix we have $Rc = c^1 u_1 + c^2 u_2 + c^3 u_3$ This equation tells us that the coordinates in the original reference frame of the vector, whose coordinates in the new reference frame are furnished by the elements of c, are provided by the elements of Rc On setting $c = R^*a$ and using the fact that $RR^* = E_3$, we have the following result

The coordinates in the original reference frame of the vector, whose coordinates in the new reference frame are furnished by the element of R^*a, are provided by the elements of a In symbols

$$a' = R^*a$$

or, equivalently (why?),

$$a = Ra'$$

Similarly, $b = Rb'$, and so the relation $b = Ma$ yields $Rb' = MRa'$ or, equivalently,

$$b' = M'a'$$

where $M' = R^*MR$ We say that the elements of M furnish the coordinates in the original reference frame of a *linear vector operator* **M** and that the coordinates of this *same* linear vector operator in the new reference frame are furnished by the elements of $M' = R^*MR$ (just as the coordinates of the vector **a** in the original reference frame are furnished by the elements of the 3×1 matrix a whereas the coor-

and $\mathbf{b} = v(b^1, b^2, b^3)$ are said to be *orthogonal* (or perpendicular) when $b^*a = 0$ (Show that the orthogonality of two complex vectors is a commutative relation, i e, that $b^*a = 0$ if, and only if, $a^*b = 0$ *Hint* $b^*a = (a^*b)^*$) A 3×3 complex matrix each of whose column vectors is of unit magnitude, these unit vectors being mutually perpendicular, is termed a *unitary* matrix, and we denote a unitary matrix by the symbol U

EXERCISES

1 Show that a 3×3 matrix U is unitary if, and only if, $U^*U = E_3$ *Note* The result of this exercise tells us that a 3×3 matrix is unitary if and only if, its reciprocal is its star, i e, the conjugate of its transpose

2 Show that a 3×3 matrix U is unitary if, and only if, $UU^* = E_3$

3 Show that every rotation matrix R is unitary

4. Show that if U is unitary det U is a complex number of unit modulus i e, that det $U = e^{i\theta}$, θ real *Hint* det U^* is the complex conjugate of det U

5 Show that if each of the column vectors of a unitary matrix is multiplied by a complex number of unit modulus (not necessarily the same for the various column vectors) the resulting matrix is still unitary

6 Show how to construct a unitary matrix whose first column vector is an arbitrary given unit vector *Hint* The second column vector $\mathbf{u}_2$ is any unit vector perpendicular to the given unit vector $\mathbf{u}_1$, and the third column vector is a unit vector whose coordinates are proportional to the *conjugates* of the coordinates of $(\mathbf{u}_1 \times \mathbf{u}_2)$

7. Show how to construct a *unimodular* unitary matrix, i e, a unitary matrix whose determinant is unity, whose first column vector is an arbitrary given unit vector *Hint* Combine Exercises 6 and 5

8 Show that the general two-dimensional unitary matrix is of the form

$$\begin{pmatrix} a & -e^{i\theta}\bar{b} \\ b & e^{i\theta}\bar{a} \end{pmatrix}$$

where θ is real and $\bar{a}a + \bar{b}b = 1$

9 Show that the general unimodular two-dimensional unitary matrix is of the form

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \quad \text{where } \bar{a}a + \bar{b}b = 1$$

10 Show that the product of two unitary matrices is unitary *Hint*

$$(U_1U_2)^*U_1U_2 = U_2^*U_1^*U_1U_2 = U_2^*U_2 = E_3$$

11. Verify that the product of two matrices of the type $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, where $\bar{a}a + \bar{b}b = 1$, is again of this type

Unimodular unitary matrices U play in the complex field the role played by rotation matrices R in the real field Thus, if a is any 3×1 complex matrix whose elements furnish the coordinates of a three-dimensional vector $\mathbf{a}$, the coordinates of this *same* vector in a second

complex rectangular Cartesian reference frame are furnished by the matrix $a' = U^*a$ where the column vectors of U furnish the coordinates, relative to the first coordinate reference frame, of unit vectors along the axes of the second coordinate reference frame. If **M** is a linear vector operator that converts, by multiplication, the complex vector **a** into a complex vector **b**,

$$\mathbf{b} = \mathbf{M}\mathbf{a},$$

we have the two equations

$$b = Ma, \qquad b' = M'a'$$

where the elements of the two 3×3 matrices M, M' furnish the *coordinates* of the linear vector operator **M** in the first and second complex rectangular Cartesian reference frames, respectively. The connection between the two different *presentations* of the *same* linear vector operator **M** is furnished by either of the two equivalent formulas

$$M = U^*M'U, \qquad M' = UMU^*$$

(prove the equivalence of these two formulas).

We are now prepared to attack the problem of determining a particular rectangular Cartesian reference frame (in general, complex) in which the presentation M' of a linear vector operator **M** (whose presentation in a given rectangular Cartesian reference frame, usually real, is a given 3×3 matrix M) is particularly simple. Let λ be a root of the cubic equation $\det (M - \lambda E_3) = 0$. Then there exists a non-zero 3×1 matrix a that satisfies the equation $Ma = \lambda a$ (why?). If λ happens to be real and if the elements of M are real the elements of a may be taken to be real, but if λ is a non-real complex number the elements of a will not all be real when the elements of M are real (for if the elements of a were real the elements of Ma would be real and we could not have $Ma = \lambda a$ since λ is, by hypothesis, non-real). In any event let u_1 be a 3×1 matrix, whose column vector u_1 is of unit magnitude, which is proportional to a (i.e., $u_1 = ka$ where k is some complex number). Construct a 3×3 unimodular unitary matrix U_1 whose first column vector is u_1. Then, on denoting by e_1 the 3×1 matrix $\begin{pmatrix}1\\0\\0\end{pmatrix}$, we have the two relations

$$Mu_1 = \lambda u_1, \qquad u_1 = U_1e_1$$

(show this), and these imply the relation $MU_1e_1 = \lambda U_1e_1$ or, equivalently,

$$U_1^*MU_1e_1 = \lambda e_1.$$

In other words, the first column of $U_1{}^*MU_1$ is $\begin{pmatrix}\lambda\\0\\0\end{pmatrix}$ Thus the complex rectangular Cartesian reference frame determined by U_1 is such that the presentation in it of the linear vector operator **M** has zeros in the second and third rows of the first column We may write this presentation of **M** in the form

$$\begin{pmatrix}\lambda & * & * \\ 0 & & \\ 0 & N & \end{pmatrix}$$

where N is a two-dimensional matrix and the stars denote numbers (complex) about whose values we can say nothing further Applying to N the argument just presented concerning M, we construct a two-dimensional unimodular unitary matrix V which is such that

$$V^*NV = \begin{pmatrix}\mu & * \\ 0 & \nu\end{pmatrix}$$

The matrix $U_2 = \begin{pmatrix}1 & 0 & 0 \\ 0 & & \\ 0 & V & \end{pmatrix}$ is a three-dimensional unimodular unitary matrix (prove this) which is such that

$$U_2{}^*\begin{pmatrix}\lambda & * & * \\ 0 & & \\ 0 & N & \end{pmatrix}U_2 = \begin{pmatrix}\lambda & * & * \\ 0 & \mu & * \\ 0 & 0 & \nu\end{pmatrix}$$

(prove this), and it follows that

$$U_2{}^*U_1{}^*MU_1U_2 = \begin{pmatrix}\lambda & * & * \\ 0 & \mu & * \\ 0 & 0 & \nu\end{pmatrix}$$

The matrix U_1U_2 is a three-dimensional unimodular unitary matrix that we denote simply by U We have, then, the following fundamentally important result

If M is any three-dimensional square matrix, there exists a three dimensional unimodular unitary matrix U which is such that U^*MU has zeros below the diagonal We term the matrix $M' = U^*MU$, which has zeros below the diagonal, the *canonical form* of the linear vector operator whose coordinates, in the original reference frame, are furnished by the elements of M

If M is such that M^* is the same as M we term it *symmetric* if its elements are real and *Hermitian* if its elements are complex. It is clear that $M' = U^*MU$ satisfies the same relation $(M')^* = M'$ (prove this) and so M' is diagonal with real diagonal elements. If the elements of M are real it follows (prove this) that the elements of U may be taken to be real (so that U is a rotation matrix). We have, then, the following important special case of our principal result:

If M is any three-dimensional symmetric matrix there exists a rotation matrix R such that R^*MR is diagonal with real diagonal elements:

$$R^*MR = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$

The three real numbers (m_1, m_2, m_3) are known as the characteristic numbers of the linear vector operator $\mathbf{M}$ whose coordinates in the original reference frame are furnished by the elements of the real symmetric matrix M.

real complex number x^*Mx is never negative (no matter what is the complex vector **x**) we say that the linear vector operator **M**, whose coordinates are furnished by the elements of M, is *non-negative* It is clear that **M** is non negative if, and only if, its characteristic numbers are non-negative (Prove this *Hint* $x'^*M'x' = m_1\bar{x}^1x^1 + m_2\bar{x}^{2'}x^{2'} + m_3x^3x^{3'}$) We now construct the 3 × 3 matrix

$$\begin{pmatrix} m_1^{1/2} & 0 & 0 \\ 0 & m_2^{1/2} & 0 \\ 0 & 0 & m_3^{1/2} \end{pmatrix}$$

where $m_1^{1/2}$ denotes the unambiguously determinate non negative square root of m_1, etc , and it is clear that the square of this matrix is the canonical form M' of M We denote it, therefore, by $(M')^{1/2}$, and it is clear that $(M')^{1/2}$ commutes with M', i e , that $(M')^{1/2}M' = M'(M')^{1/2}$, since both M' and $(M')^{1/2}$ are diagonal matrices On setting

$$M^{1/2} = U(M')^{1/2}U^*$$

(so that the elements of $M^{1/2}$ are the coordinates in our original reference frame of the linear vector operator $\mathbf{M}^{1/2}$ whose coordinates, in the reference frame in which the coordinates of **M** are furnished by the elements of M', are furnished by the elements of $(M')^{1/2}$), it is clear that $\mathbf{M}^{1/2}$ is a non negative linear vector operator (why?) and that the square of $\mathbf{M}^{1/2}$ is **M** (prove this) It is also clear that $\mathbf{M}^{1/2}$ commutes with **M**, i e , that $\mathbf{M}^{1/2}\mathbf{M} = \mathbf{M}\mathbf{M}^{1/2}$ (prove this)

If A is any 3 × 3 matrix it is clear that the 3 × 3 matrix M defined by $M = A^*A$ furnishes the coordinates of a non negative linear vector operator In fact, $M^* = M$ (prove this) and $x^*Mx = x^*A^*Ax = (Ax)^*Ax$ is never negative, being the squared magnitude of the vector whose coordinates are furnished by the elements of the 3 × 1 matrix Ax If the matrix A is non singular, **M** is not only non-negative but also *positive,* i e , x^*Mx is positive save when x is the zero 3 × 1 matrix (in which event x^*Mx is evidently zero) In this case M is non singular (why?), and this implies that $M^{1/2}$ is non singular (why?) Denoting the reciprocal of $M^{1/2}$ by $M^{-1/2}$, we can easily see that the matrix $AM^{-1/2}$ is unitary In fact, $(AM^{-1/2})^*(AM^{-1/2}) = M^{-1/2}A^*AM^{-1/2} = M^{-1/2}MM^{-1/2} = (M^{-1/2})^2M = M^{-1}M = E_3$ On denoting the unitary matrix $AM^{-1/2}$ by U, we have $A = UM^{1/2}$ We obtain, in this way, the following fundamental result

Every non singular 3 × 3 matrix A whose elements are complex numbers may be written in the form

$$A = UM^{1/2}$$

where U is a unitary 3×3 matrix and $M^{1/2}$ is a positive (and, hence, Hermitian) 3×3 matrix.

If the elements of A are real, $M = A^*A$ may be transformed into its canonical form by a rotation matrix (why?), and so the elements of $M^{-1/2}$ are real (why?). Hence the elements of $U = AM^{-1/2}$ are real. If $\det A > 0$ it follows (since $\det M^{-1/2} > 0$) that $\det U > 0$ and so U is a rotation matrix. We have, then, the following particular instance of our principal result:

If A is a non-singular 3×3 matrix, with real elements, whose determinant is positive, A may be written in the form

$$A = RM^{1/2}$$

where R is a rotation matrix and $M = A^*A$; $M^{1/2}$ is a symmetric matrix whose characteristic numbers $m_1^{1/2}$, $m_2^{1/2}$, $m_3^{1/2}$ are all positive (m_1, m_2, m_3 being the positive characteristic numbers of $M = A^*A$).

This result is of fundamental importance in the theory of elasticity.

6. Conclusion

If one has mastered fairly well the contents of this introductory chapter he is in possession of all the matrix theory required in this treatment of the finite deformations of an elastic solid. For an introductory study at a level suited for senior high school or beginning college students the reader is referred to the first five chapters of *Analytic Geometry*,[1] where analytic geometry is treated from the very beginning from the point of view of vectors and matrices. More advanced matrix theory is presented in the first two chapters of *Introduction to Applied Mathematics*,[2] and a treatment of those parts of the theory of matrices which are of importance in group theory is given in *Theory of Group Representations*.[3] Try to become familiar with matrices and to use them regularly as a helpful tool. Do not fall into the error of regarding them as a complicated device invented by mathematicians to make the theory of elasticity harder than it actually is.

2. Write down the most general 3×3 matrix that commutes with the diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_1 \end{pmatrix}$$

where $d_2 \neq d_1$

Answer $A = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$ where the stars indicate elements about whose values we can say nothing

3 Calculate the matrix $M = A^*A$ where A is the Jacobian matrix $\dfrac{(x, y, z)}{(a, b, c)}$ of the shear $x = a + kc$, $y = b$ $z = c$ (k constant)

4 Repeat Exercise 3 for the torsion $x = r\cos(\theta + kc) = a\cos kc - b\sin kc$, $y = r\sin(\theta + kc) = a\sin kc + b\cos kc$, $z = c$ (k constant)

5 Determine the axis and angle of the rotation whose matrix is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Answer The line from (0, 0, 0) to (1, 1, 1), $\dfrac{2\pi}{3}$.

2

THE SPECIFICATION OF STRAIN

1. The strain matrix

We consider two states of a three-dimensional deformable medium:

(1) The *initial* or *unstrained* state. We denote the rectangular Cartesian coordinates of a typical particle of the medium, when in the unstrained state, with respect to any convenient reference frame, by (a, b, c).

(2) The *final* or *strained* or *deformed* state. We denote the rectangular Cartesian coordinates of the particle, whose coordinates when the medium was in the initial unstrained state were (a, b, c), by (x, y, z). The reference frame with respect to which (x, y, z) are measured need not be the same (although it is usually convenient to have it the same) as the reference frame with respect to which (a, b, c) are measured.

If we consider a collection of particles occupying, when the medium is unstrained, a certain volume V_a, a, b, and c will each be functions of three independent variables (α, β, γ) which are such that $\det \dfrac{(a, b, c)}{(\alpha, \beta, \gamma)} \neq 0$. We suppose the independent variables, or parameters, α, β, and γ so ordered that $\det \dfrac{(a, b, c)}{(\alpha, \beta, \gamma)} > 0$, and we denote by dV_a the product $\det \dfrac{(a, b, c)}{(\alpha, \beta, \gamma)}\, d\alpha\, d\beta\, d\gamma$. dV_a is the *initial element of volume* (i.e., the element of volume of the medium when it is in its unstrained state), and we may, by adopting the alternating rule of multiplication (what is this?) of the differentials $d\alpha$, $d\beta$, $d\gamma$, write it in the form $da\, db\, dc$:

$$dV_a = da\, db\, dc = \det \frac{(a, b, c)}{(\alpha, \beta, \gamma)}\, d\alpha\, d\beta\, d\gamma.$$

The coordinates (x, y, z) of the final position of the particle whose initial coordinates are (a, b, c) are assumed to be differentiable functions of (a, b, c), and it is assumed further that the determinant of the

Jacobian matrix $\dfrac{(x, y, z)}{(a, b, c)}$ is positive. Then the final element of volume is given by the formula

$$dV_x = dx\,dy\,dz = \det \frac{(x, y, z)}{(a, b, c)}\,dV_a = (\det J)\,dV_a$$

where J denotes the Jacobian matrix $\dfrac{(x, y, z)}{(a, b, c)}$. If ρ_a and ρ_x denote, respectively, the initial and final mass densities, the principle of conservation of mass assures us that

$$\rho_x\,dV_x = \rho_a\,dV_a$$

and so the compression ratio $\dfrac{\rho_x}{\rho_a}$ is furnished by the formula

$$\frac{\rho_x}{\rho_a} = \frac{1}{\det J}$$

Let us now consider a collection of particles that lie initially on a curve C_a. Then a, b, and c are functions of a single independent variable or parameter α, and the matrix element of arc of C_a is the 3×1 matrix

$$da = \begin{pmatrix} da \\ db \\ dc \end{pmatrix}$$

The scalar element of arc ds_a of C_a is the positive square root of $(da)^*da$

$$(ds_a)^2 = (da)^*\,da$$

The particles of the deformable medium that originally lay on the curve C_a lie, after the deformation, on a curve C_x, and the matrix element of arc of C_x is the 3×1 matrix

$$dx = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = J\,da$$

The scalar element of arc ds_x of C_x is the positive square root of $(dx)^*\,dx$

$$(ds_x)^2 = (dx)^*dx = (da)^*J^*J\,da$$

If ds_x is the same as ds_a for every curve C_a (i e , for an arbitrary 3×1 matrix da) all curve lengths are unchanged by the "deformation" so that, in particular straight lines are "deformed" into straight lines

(why?). This means that x, y, and z are linear functions of a, b, and c (why?) and so J is a constant matrix. Since this constant matrix satisfies the relation

$$(da)^*J^*J\,da = (da)^*\,da$$

for arbitrary choices of the 3×1 matrix da we must have $J^*J = E_3$ (prove this). Since $\det J > 0$ it follows that J is a rotation matrix. Thus the transformation from the initial coordinates (a, b, c) of any particle of the medium to the final coordinates (x, y, z) of the same particle is merely that involved in the transformation of coordinates from one rectangular Cartesian reference frame to another. In other words, the change from the initial to the final position of the medium is a mere *rigid displacement* and no *deformation* or change of shape of the medium (or of any portion of the medium) is involved. When J is not a rotation matrix we use the difference between J^*J and E_3 as a measure of the deformation of the medium in the neighborhood of the point (a, b, c). We set

$$\tfrac{1}{2}(J^*J - E_3) = \eta$$

and we say that the 3×3 symmetric matrix η measures the deformation of the elastic medium at the point (a, b, c). This matrix η is known as the *strain matrix*, and it satisfies the relation

$$(ds_x)^2 - (ds_a)^2 = 2(da)^*\eta\,da$$

(prove this). The use of the factor $\frac{1}{2}$ in the definition of η or, equivalently, of the factor 2 in the relation just written, is due to the fact that this relation involves the difference of the *squares* of ds_x and ds_a. If the strain or deformation is small enough so that we can replace $ds_x + ds_a$ by $2ds_a$, $(ds_x)^2 - (ds_a)^2$ becomes $2(ds_x - ds_a)(ds_a)$ (why cannot we replace $ds_x - ds_a$ by zero?), and we have the approximate relation

$$\frac{ds_x - ds_a}{ds_a} = \left(\frac{da}{ds_a}\right)^* \eta\,\frac{da}{ds_a}.$$

The column vector of the 3×1 matrix $\dfrac{da}{ds_a}$ is a unit vector; if this 3×1 matrix is denoted by u, the quadratic form $u^*\eta u$ in the elements of u measures, approximately, the *relative magnifications* $\dfrac{(ds_x - ds_a)}{ds_a}$ in the neighborhood of the point (a, b, c). It is clear from the relation $(ds_x)^2 - (ds_a)^2 = 2(da)^*\eta\,da$ that the quadratic form $u^*\eta u$ measures exactly $\dfrac{1}{2}\dfrac{(ds_x)^2 - (ds_a)^2}{(ds_a)^2}$ (prove this).

The most direct way of calculating η for any given deformation is first to set up $J = \dfrac{(x, y, z)}{(a, b, c)}$ and then to calculate the symmetric matrix J^*J. The diagonal elements of J^*J are the squared magnitudes of the column vectors of J, and each non-diagonal element of J^*J is a scalar product of two of these column vectors. After J^*J has been evaluated, subtract E_3 from it, i.e., diminish each of the diagonal elements of J^*J by unity; one half the resulting matrix is the desired strain matrix.

Example Calculate η for the shear $x = a + kc$, $y = b$, $z = c$, where k is a constant. Here

$$J = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad J^*J = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ k & 0 & 1 + k^2 \end{pmatrix}$$

and so

$$\eta = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & k^2 \end{pmatrix}$$

If k is so small that its square may be neglected we say that the shear is *infinitesimal*; thus the strain matrix for an infinitesimal shear parallel to the a axis in the ac-plane equals

$$\tfrac{1}{2}\begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$$

EXERCISES

1. Calculate the strain matrix η for the uniform compression or dilatation $x = a + ka$, $y = b + kb$, $z = c + kc$, k a constant. *Note*. This is a compression if $k < 0$ and a dilatation if $k > 0$.

Answer $\eta = (k + \frac{1}{2}k^2)E_3$

2. What is the strain matrix for an infinitesimal uniform compression or dilatation?

Answer $\eta = kF_3$

3. Calculate the strain matrix η for the torsion $x = r\cos(\theta + kc) = a\cos kc - b\sin kc$, $y = r\sin(\theta + kc) = a\sin kc + b\cos kc$, $z = c$, k a constant. What does η reduce to when the torsion is infinitesimal?

Answer $\tfrac{1}{2}\begin{pmatrix} 0 & 0 & -kb \\ 0 & 0 & ka \\ -kb & ka & k^2(a^2 + b^2) \end{pmatrix}$; $\tfrac{1}{2}\begin{pmatrix} 0 & 0 & -kb \\ 0 & 0 & ka \\ -kb & ka & 0 \end{pmatrix}$

4. Calculate the strain matrix η for the simple tension and cross-sectional contraction of a rod defined by $x = a - \sigma ka$, $y = b - \sigma kb$, $z = c + kc$, where k and σ

are constants. *Note.* The constant σ is known as Poisson's ratio (after S. D. Poisson [1781–1840], a French mathematician).

Answer. $\eta = \begin{pmatrix} -\sigma k + \frac{1}{2}\sigma^2 k^2 & 0 & 0 \\ 0 & -\sigma k + \frac{1}{2}\sigma^2 k^2 & 0 \\ 0 & 0 & k + \frac{1}{2}k^2 \end{pmatrix}.$

5. What does the strain matrix of Exercise 4 reduce to when the strain is infinitesimal, i.e., when the square of k may be neglected?

Answer. $\eta = \begin{pmatrix} -\sigma k & 0 & 0 \\ 0 & -\sigma k & 0 \\ 0 & 0 & k \end{pmatrix}.$

2. The behavior of the strain matrix under transformations of the initial and final coordinate reference frames

A rotation of the initial coordinate reference frame, i.e., of the rectangular Cartesian reference frame with respect to which the coordinates of a typical particle of the elastic medium are (a, b, c), is described by the formula

$$a \to a' = R^*a$$

where the coordinates with respect to the old reference frame of unit vectors along the axes of the new reference frame are furnished by the column vectors of R. We understand by the notation $a \to a' = R^*a$ that the new initial coordinates of the particle, whose original initial coordinates were furnished by the elements of the 3×1 matrix a, are furnished by the elements of the 3×1 matrix R^*a. It follows from the relation $a' = R^*a$ that $a = Ra'$ and, hence, that $\dfrac{(a, b, c)}{(a', b', c')} = R$. Since $\dfrac{(x, y, z)}{(a', b', c')} = \dfrac{(x, y, z)}{(a, b, c)} \dfrac{(a, b, c)}{(a', b', c')}$ we have the following result:

Under the transformation $a \to a' = R^*a$ of the initial rectangular Cartesian reference frame, $J \to J' = JR$, the final coordinate reference frame being unchanged.

EXERCISES

1. Show that under the transformation $x \to x' = R^*x$ of the final rectangular Cartesian frame $J \to J' = R^*J$.

2. Show that under the simultaneous transformations $a \to a' = R^*a$, $x \to x' = R^*x$ of both the initial and the final rectangular Cartesian reference frames $J \to J' = R^*JR$.

Since $M = J^*J$ and since the transformation $J \to J' = JR$ implies the transformation $J^* \to J^{*\prime} = R^*J^*$ (why?), it follows that $M \to M' = R^*MR$. Since the strain matrix $\eta = \frac{1}{2}(M - E_3)$, this fact yields the following important result:

Under the transformation $a \to a' = R^*a$ of the initial rectangular Cartesian reference frame $\eta \to \eta' = R^*\eta R$

In other words, the strain matrix η is sensitive to the initial rectangular Cartesian reference frame There exists (at each point of the medium) at least one particular initial reference frame in which η appears in canonical (= diagonal) form We term the axes of any such particular initial reference frame *principal axes of strain* and we write the strain matrix η when referred to a reference frame whose axes are principal axes of strain, as follows

$$\begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}$$

Note carefully that, unless η is a constant matrix, the principal axes of strain will vary from point to point of the medium Note further that if R is a rotation matrix that transforms η into its canonical form, so that $R^*\eta R = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}$ we have $\eta R = R \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}$ Hence the column vectors of R are characteristic vectors of η (why?) The easiest way to determine R is to determine the characteristic vectors of η It frequently happens that one of these is relatively easy to determine (the determination of the others being comparatively difficult) The best thing to do in this event is to construct a preliminary rotation matrix R_1 whose first column vector has the direction of the easily determined characteristic vector of η Then $R_1{}^*\eta R_1$ will be of the form

$$\begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

and our problem becomes the comparatively simple one of a plane strain whose matrix is indicated by $\begin{matrix} * & * \\ * & * \end{matrix}$

In contrast with its behavior with respect to a change of the initial rectangular Cartesian reference frame the strain matrix η is *insensitive* to a change of the final rectangular Cartesian reference frame In fact, under the transformation $x \to x' = R^*x$, $J \to J' = R^*J$, $J^* \to J^{*\prime} = J^*R$, $M \to M' = J^*RR^*J = M$ Hence $\eta \to \eta' = \eta$ (why?) Thus the elements of the strain matrix η are *invariants* under transformation of the final rectangular Cartesian reference frame, to

emphasize this point we denote these elements by the symbols η_{aa}, η_{ab}, etc.:

$$\eta = \begin{pmatrix} \eta_{aa} & \eta_{ab} & \eta_{ac} \\ \eta_{ba} & \eta_{bb} & \eta_{bc} \\ \eta_{ca} & \eta_{cb} & \eta_{cc} \end{pmatrix}.$$

Note that the first of the two subscripts tells the row and the second the column; thus η_{bc} is the element in the second row and third column of η.

Remark 1. We use η_{aa}, η_{ab}, etc., instead of $\eta_a{}^a$, $\eta_b{}^a$, etc., since η is not the presentation of a linear vector operator but rather the matrix of coefficients of a quadratic form $u^*\eta u$ where u is the 3×1 matrix element of arc $\dfrac{da}{ds_a}$ of C_a whose column vector is of unit magnitude. The fact that the strain matrix η is symmetric is expressed by the relations

$$\eta_{cb} = \eta_{bc}, \qquad \eta_{ac} = \eta_{ca}, \qquad \eta_{ba} = \eta_{ab}.$$

Remark 2. The *insensitiveness* of η to a change in the final rectangular Cartesian reference frame may be expressed by the statement that the strain matrix is *unaffected* if the given deformation is *followed* by a rigid rotation of the medium; thus both J and R^*J yield the same η. The *sensitiveness* of η to a change in the initial rectangular Cartesian reference frame may be expressed by the statement that the strain matrix is *affected* if the given deformation is *preceded* by a rigid rotation of the medium. Thus J and JR yield different strain matrices.

EXERCISE

3. Show that the strain matrix η transforms under the simultaneous transformation $a \to a' = R^*a$, $x \to x' = R^*x$ of the initial and final rectangular Cartesian reference frames according to the formula $\eta \to \eta' = R^*\eta R$. *Hint.* η is insensitive to the transformation $x' = R^*x$.

Example 1: Determine the principal axes of the plane shear $x = a + kb$, $y = b$, k constant. Here $J = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and so $\eta = \frac{1}{2}\begin{pmatrix} 0 & k \\ k & k^2 \end{pmatrix}$. Setting $R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ (so that the new initial reference frame is obtained by rotating the old initial reference frame through the angle α), we have

$$2\eta R = \begin{pmatrix} k\sin\alpha & k\cos\alpha \\ k\cos\alpha + k^2\sin\alpha & -k\sin\alpha + k^2\cos\alpha \end{pmatrix},$$

$$2R^*\eta R = \begin{pmatrix} k\sin 2\alpha + k^2\sin^2\alpha & k\cos 2\alpha + \frac{1}{2}k^2\sin 2\alpha \\ k\cos 2\alpha + \frac{1}{2}k^2\sin 2\alpha & -k\sin 2\alpha + k^2\cos^2\alpha \end{pmatrix}.$$

In order that $R^*\eta R$ be diagonal we must have $\cot 2\alpha = -\frac{k}{2}$ or, equivalently, $\sin 2\alpha = \frac{2}{(4 + k^2)^{1/2}}$ (α lying in the interval $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$, if $k > 0$) On eliminating α, we find that $\eta_1 = \frac{k}{(4 + k^2)^{1/2}} + \frac{k^2}{4} + \frac{k^3}{4(4 + k^2)^{1/2}}$, $\eta_2 = \frac{-k}{(4 + k^2)^{1/2}} + \frac{k^2}{4} - \frac{k^3}{4(4 + k^2)^{1/2}}$ *Note* In the infinitesimal theory of plane shear (in which k^2 is neglected) the angle α (being furnished by the formula $\sin 2\alpha = 1$) turns out to be $\frac{\pi}{4}$ and so is independent of k To this degree of approximation $\eta_1 = \frac{k}{2}$, $\eta_2 = \frac{-k}{2}$ The next approximation to η_1 and η_2 (in which we neglect k^3 and higher powers of k but not k^2) is $\eta_1 = \frac{k}{2} + \frac{k^2}{4}$, $\eta_2 = \frac{-k}{2} + \frac{k^2}{4}$ The formula $\cot 2\alpha = \frac{-k}{2}$ which furnishes the exact angle of rotation is so simple that there is no point in *approximating* α, if we write $\alpha = \frac{\pi}{4} + \beta$ we have $\tan 2\beta = \frac{k}{2}$ and so the exact angle of rotation may be written in the convenient form $\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{k}{2}$

Example 2 Determine the principal axes of the torsion $x = r\cos(\theta + kc)$ $y = r\sin(\theta + kc)$, $z = c$, k constant Here $x = a\cos kc - b\sin kc$ $y = a\sin kc + b\cos kc$ $z = c$, and so

$$J = \begin{pmatrix} \cos kc & -\sin kc & -ky \\ \sin kc & \cos kc & kx \\ 0 & 0 & 1 \end{pmatrix} \qquad \eta = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & -kb \\ 0 & 0 & ka \\ -kb & ka & k^2r^2 \end{pmatrix}$$

Since $\det \eta = 0$, one of the characteristic numbers of η is 0, and it is clear that $v(a, b, 0)$ is an associated characteristic vector Setting up the preliminary rotation matrix

$$R_1 = \begin{pmatrix} \frac{a}{r} & \frac{-b}{r} & 0 \\ \frac{b}{r} & \frac{a}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ we have } 2\eta R_1 = \begin{pmatrix} 0 & 0 & -kb \\ 0 & 0 & ka \\ 0 & kr & k^2r^2 \end{pmatrix}$$

$2R_1^*\eta R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & kr \\ 0 & kr & k^2r^2 \end{pmatrix}$ and we have only to deal with the plane shear (in the $b'c$-plane) whose strain matrix is $\frac{1}{2}\begin{pmatrix} 0 & kr \\ kr & k^2r^2 \end{pmatrix}$. Thus if we start with an initial reference frame whose first and second axes have the directions of plane polar coordinate lines at the point (a, b) in the (a, b)-plane the principal axes of the torsion are obtained by rotating these axes around the radial direction through the angle $\frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{kr}{2}$. The canonical form of the torsion strain matrix is furnished by the formulas $\eta_1 = 0$, $\eta_2 = \frac{kr}{(4 + k^2r^2)^{1/2}} + \frac{k^2r^2}{4} + \frac{k^3r^3}{4(4 + k^2r^2)^{1/2}}$, $\eta_3 = -\frac{kr}{(4 + k^2r^2)^{1/2}} + \frac{k^2r^2}{4} - \frac{k^3r^3}{4(4 + k^2r^2)^{1/2}}$. If we are dealing with a cylinder of length l which is twisted through a quarter turn at $c = l$ we have $kl = \frac{\pi}{4}$ and so $kr = \frac{\pi}{4}\frac{r}{l}$. Thus the approximation furnished by the infinitesimal theory will not be valid at the surface of the cylinder (for a torsion of this magnitude) unless the ratio of the radius of the cylinder to its length is so small that its square may be neglected.

If we use the *same* reference frame for the initial and final positions of the medium we may introduce the *displacement vector* $v(U, V, W)$ whose coordinates are defined by the formulas

$$U = x - a, \qquad V = y - b, \qquad W = z - c.$$

It follows that $J = E_3 + K$ where K is the Jacobian matrix $\frac{(U, V, W)}{(a, b, c)}$ and so $M = J^*J = E_3 + K^* + K + K^*K$. Hence

$$\eta = \tfrac{1}{2}(M - E_3) = \tfrac{1}{2}(K^* + K) + \tfrac{1}{2}K^*K.$$

In the *infinitesimal* theory of strain, K is regarded as an infinitesimal matrix, i.e., as a matrix whose elements are so small that their squares and products are negligible. To this degree of approximation, then,

$$\eta = \tfrac{1}{2}(K^* + K) = \begin{pmatrix} U_a & \frac{1}{2}(V_a + U_b) & \frac{1}{2}(U_c + W_a) \\ \frac{1}{2}(V_a + U_b) & V_b & \frac{1}{2}(W_b + V_c) \\ \frac{1}{2}(U_c + W_a) & \frac{1}{2}(W_b + V_c) & W_c \end{pmatrix}.$$

To obtain the exact expressions for the elements of η we must add $\frac{1}{2}K^*K$ to $\frac{1}{2}(K^* + K)$; remembering that the elements of K^*K are

the squared magnitudes and the scalar products of the column vectors of K, we obtain

$$\eta_{aa} = U_a + \tfrac{1}{2}(U_a^2 + V_a^2 + W_a^2),$$
$$\eta_{bb} = V_b + \tfrac{1}{2}(U_b^2 + V_b^2 + W_b^2),$$
$$\eta_{cc} = W_c + \tfrac{1}{2}(U_c^2 + V_c^2 + W_c^2),$$
$$\eta_{bc} = \tfrac{1}{2}(W_b + V_c) + \tfrac{1}{2}(U_bU_c + V_bV_c + W_bW_c),$$
$$\eta_{ca} = \tfrac{1}{2}(U_c + W_a) + \tfrac{1}{2}(U_cU_a + V_cV_a + W_cW_a),$$
$$\eta_{ab} = \tfrac{1}{2}(V_a + U_b) + \tfrac{1}{2}(U_aU_b + V_aV_b + W_aW_b)$$

3. The invariants of the strain matrix

We have seen that the strain matrix η is sensitive to a rotation of the initial rectangular Cartesian reference frame, under the transformation $a \to a' = R^*a$, $\eta \to \eta' = R^*\eta R$ It follows that, no matter what the number λ is, $\eta' - \lambda E_3 = R^*(\eta - \lambda E_3)R$ (prove this) Hence det $(\eta' - \lambda E_3) =$ det $(\eta - \lambda E_3)$ (why?) If we denote by I_1, I_2, and I_3, respectively, the trace of η, the trace of co η and the determinant of η, we have

$$\det(\eta - \lambda E_3) = I_3 - I_2\lambda + I_1\lambda^2 - \lambda^3$$

(prove this) If I_1', I_2', I_3' are, respectively, $Tr\, \eta'$, Tr co η', and det η' it follows that, no matter what the number λ is,

$$I_3' - I_2'\lambda + I_1'\lambda^2 - \lambda^3 = I_3 - I_2\lambda + I_1\lambda^2 - \lambda^3$$

Hence (why?)

$$I_3' = I_3 \quad I_2' = I_2, \; I_1' = I_1$$

Thus, although η is sensitive to a rotation of the initial rectangular Cartesian reference frame, the three functions I_1, I_2, and I_3 of the elements of η (or, as we shall simply say, of η) are insensitive to any such rotation We term them the three *invariants* of η They are furnished by the formulas $I_1 = \eta_{aa} + \eta_{bb} + \eta_{cc}$, in words, I_1 is the sum of the three diagonal elements of η,

$$I_2 = \begin{vmatrix} \eta_{bb} & \eta_{bc} \\ \eta_{cb} & \eta_{cc} \end{vmatrix} + \begin{vmatrix} \eta_{cc} & \eta_{ca} \\ \eta_{ac} & \eta_{aa} \end{vmatrix} + \begin{vmatrix} \eta_{aa} & \eta_{ab} \\ \eta_{ba} & \eta_{bb} \end{vmatrix},$$

in words, I_2 is the sum of the three two-rowed diagonal minors of η,

$$I_3 = \det \eta,$$

in words, I_3 is the determinant of η. On writing η in its canonical (= diagonal) form, we see that

$$I_1 = \eta_1 + \eta_2 + \eta_3, \qquad I_2 = \eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2, \qquad I_3 = \eta_1\eta_2\eta_3.$$

If η is an infinitesimal matrix I_1, I_2, and I_3 are, respectively, infinitesimals of (at least) the first, second, and third orders.

It is easy to express the compression ratio $\dfrac{\rho_x}{\rho_a}$ in terms of the invariants I_1, I_2, and I_3 of η. In fact, $\rho_a = \rho_x (\det J)$ and $\det (E_3 + 2\eta) = \det M = \det J^*J = (\det J)^2$ and so

$$\left(\frac{\rho_a}{\rho_x}\right)^2 = \det (E_3 + 2\eta) = 1 + 2I_1 + 4I_2 + 8I_3.$$

Hence

$$\frac{\rho_x}{\rho_a} = (1 + 2I_1 + 4I_2 + 8I_3)^{-1/2}.$$

The approximation to this exact formula which is furnished by the infinitesimal theory is

$$\frac{\rho_x}{\rho_a} = (1 + 2I_1)^{-1/2} = 1 - I_1.$$

To this degree of approximation, $I_1 = \dfrac{\rho_a - \rho_x}{\rho_a}$ is the relative decrease in density or, equivalently (in view of the principle of conservation of mass), the relative increase in (local) volume.

EXERCISES

1. Show that for the uniform dilatation (or compression) $x = (1 + k)a$, $y = (1 + k)b$, $z = (1 + k)c$, k constant, $\eta = (k + \frac{1}{2}k^2)E_3$, and calculate I_1, I_2, and I_3.
Answer. $I_1 = 3(k + \frac{1}{2}k^2)$, $I_2 = 3(k + \frac{1}{2}k^2)^2$, $I_3 = (k + \frac{1}{2}k^2)^3$.

2. Verify that, for the uniform dilatation (or compression) of Exercise 1, $I_2 = \frac{1}{3}I_1^2$, $I_3 = \frac{1}{27}I_1^3$.

3. Show that if the relations of Exercise 2 are satisfied the strain matrix is a scalar matrix, i.e., a multiple of E_3. *Hint.* The cubic equation that determines the characteristic numbers of η is $(\lambda - \frac{1}{3}I_1)^3 = 0$.

4. Show that two deformations having the same strain matrix have their Jacobian matrices J_1 and J_2 connected by the relation $J_2 = RJ_1$ where R is a rotation matrix. *Hint.* $J_1 = R_1M^{1/2}$, $J_2 = R_2M^{1/2}$, and so $J_2 = RJ_1$ where $R = R_2R_1^*$.

5. Show that the compression ratio for the uniform dilatation (or compression) of Exercise 1 is furnished by the formula $\dfrac{\rho_x}{\rho_a} = (1 + k)^{-3}$.

6 Show that for the uniform dilatation (or compression) of Exercise 1 the strain matrix η is the product of E_3 by $\frac{1}{2}\left(\frac{\rho_x}{\rho_a}\right)^{-2/3} - 1$ *Hint* $\det M = (1 + 2e)^3$ where $\eta = eE_3$

7 Calculate the invariants of the shear $x = a + kc$ $y = b$ $z = c$ and determine the relations between them

Answer $I_1 = \frac{1}{2}k^2$ $I_2 = -\frac{1}{4}k^2$ $I_3 = 0$ $I_2 = -\frac{1}{2}I_1$ $I_3 = 0$

8 Repeat Exercise 7 for the torsion $x = r\cos(\theta + kc) = a\cos kc - b\sin kc$ $y = r\sin(\theta + kc) = a\sin kc + b\cos kc$ $z = c$ and verify that this is at any point $(a\ b\ c)$ a shear whose coefficient is $kr = k(a^2 + b^2)^{1/2}$

Answer $I_1 = \frac{1}{2}k^2r^2$ $I_2 = -\frac{1}{4}k^2r^2$ $I_3 = 0$

4 The compatibility relations between the elements of the strain matrix and their derivatives

Suggestion The mathematical computations involved in this section are quite formidable for a reader who is not well trained in tensor analysis One may safely omit this section in a first reading at some later time when the reader has leisure and wants to understand what is behind the *compatibility relations* of the infinitesimal theory of elasticity he may return and work through the computations of this section

The basic relation $(dx)^* dx = (da)^* M\, da$ tells us that the quadratic form with coefficient matrix M in the elements of the 3×1 matrix da appears when written as a quadratic form in the elements of the 3×1 matrix dx as a quadratic form with the coefficient matrix E_3 whose elements are *constants* In the terminology of metric differential geometry this constancy of the elements of the coefficient matrix of the quadratic form $(dx)^* dx$ is expressed as follows

The curvature tensor of the three-dimensional metric space the coordinates of whose metrical tensor in the $(a\ b\ c)$ system of coordinates are furnished by the elements of M is the zero tensor

This fact furnishes six relations involving the elements of M and their first- and second order derivatives with respect to $(a\ b\ c)$ or, equivalently since $M = E_3 + 2\eta$ six relations involving the elements of η and their first and second-order derivatives with respect to $(a\ b\ c)$ These are the (exact) *compatibility relations* between the elements of η and their first and second derivatives with respect to $(a\ b\ c)$ If η is an infinitesimal matrix and we neglect in the (exact) compatibility relations all products of an element of η or of any derivative of an element of η by an element of η or any derivative of an element of η we obtain the compatibility relations of the infinitesimal theory of elasticity

A knowledge of tensor analysis will not be assumed here and the

discussion will simply furnish that portion (and no more) of tensor analysis which is necessary for the purpose. Following the notation of tensor analysis we use (a^1, a^2, a^3) instead of (a, b, c) and (x^1, x^2, x^3) instead of (x, y, z). The double occurrence (once below and once above) of a Greek label in a term indicates summation, with respect to that label, over the range 1, 2, 3. For example, if x^r is a typical element of a 3×1 matrix x and if y_s is a typical element of a 1×3 matrix y, the matrix product yx is denoted by the symbol $y_\alpha x^\alpha$ (or, equivalently, $y_\beta x^\beta$, $y_\gamma x^\gamma$, etc.). In this notation the fundamental relation $(dx)^* \, dx = (da)^* M \, da$ may be written as

$$m_{\alpha\beta} a^\alpha{}_{,r} a^\beta{}_{,s} = 1, \text{ if } s = r,$$
$$= 0, \text{ if } s \neq r.$$

Here m_{pq} is a typical element (the element in the pth row and qth column) of M and the comma in the symbols $a^\alpha{}_{,r}$, $a^\beta{}_{,s}$ denotes differentiation; $a^p{}_{,r}$ is the derivative of a^p with respect to x^r. The occurrence of the two Greek letters α and β in the symbol $m_{\alpha\beta} a^\alpha{}_{,r} a^\beta{}_{,s}$ indicates a *double* summation (one summation on α and one on β) so that $m_{\alpha\beta} a^\alpha{}_{,r} a^\beta{}_{,s}$ is the sum of nine terms (write down five of these terms). We now differentiate the relation $m_{\alpha\beta} a^\alpha{}_{,r} a^\beta{}_{,s} =$ constant with respect to any one of the x's, say x^t. In doing this differentiation we remember that the elements of M are given as functions of the a's so that the derivative of m_{pq} with respect to x^t is $m_{pq,\gamma} a^\gamma{}_{,t}$ where $m_{pq,r}$ denotes the derivative of m_{pq} with respect to a^r. We obtain

$$m_{\alpha\beta,\gamma} a^\alpha{}_{,r} a^\beta{}_{,s} a^\gamma{}_{,t} + m_{\alpha\beta} a^\alpha{}_{,rt} a^\beta{}_{,s} + m_{\alpha\beta} a^\alpha{}_{,r} a^\beta{}_{,st} = 0$$

where $a^p{}_{,rt}$, for example, denotes the second derivative of a^p with respect to x^r and x^t (so that $a^p{}_{,rt} = a^p{}_{,tr}$ [why?]). We solve this system of equations for the second derivatives $a^p{}_{,rt}$ as follows: First interchange r and t and rewrite the triple summation $m_{\alpha\beta,\gamma} a^\alpha{}_{,t} a^\beta{}_{,s} a^\gamma{}_{,r}$ in the equivalent form $m_{\gamma\beta,\alpha} a^\alpha{}_{,r} a^\beta{}_{,s} a^\gamma{}_{,t}$ (why are these two triple summations the same?). We obtain

$$m_{\gamma\beta,\alpha} a^\alpha{}_{,r} a^\beta{}_{,s} a^\gamma{}_{,t} + m_{\alpha\beta} a^\alpha{}_{,rt} a^\beta{}_{,s} + m_{\alpha\beta} a^\alpha{}_{,t} a^\beta{}_{,rs} = 0.$$

Interchanging s and t in our original relation and rewriting similarly our triple summation, we obtain

$$m_{\alpha\gamma,\beta} a^\alpha{}_{,r} a^\beta{}_{,s} a^\gamma{}_{,t} + m_{\alpha\beta} a^\alpha{}_{,rs} a^\beta{}_{,t} + m_{\alpha\beta} a^\alpha{}_{,r} a^\beta{}_{,st} = 0.$$

On subtracting our original relation from the sum of the two relations obtained in this way and denoting by $\Gamma_{pq;r}$ the combination $\frac{1}{2}(m_{pr,q} +$

$m_{qr\ p} - m_{pq\ r}$) of the first derivatives of the elements of M, we obtain

$$\Gamma_{\alpha\beta\ \gamma} a^{\alpha}{}_{r} a^{\beta}{}_{,s} a^{\gamma}{}_{t} + m_{\alpha\beta} a^{\alpha}{}_{rs} a^{\beta}{}_{t} = 0$$

(in deriving which we have used the fact that the two double sums $m_{\alpha\beta} a^{\alpha}{}_{t} a^{\beta}{}_{rs}$ and $m_{\alpha\beta} a^{\alpha}{}_{rs} a^{\beta}{}_{t}$ are the same (why?) (*Hint* The matrix M is symmetric) (Note that $\Gamma_{qp\ r} = \Gamma_{pq\ r}$) This relation can be written in the equivalent form

$$(\Gamma_{\alpha\beta\ \gamma} a^{\alpha}{}_{r} a^{\beta}{}_{s} + m_{\alpha\gamma} a^{\alpha}{}_{rs}) a^{\gamma}{}_{t} = 0$$

On multiplying this relation by $x^{t}{}_{p}$ (the derivative of x^{t} with respect to a^{p}) and summing with respect to t, we obtain, since $a^{j}{}_{r} x^{r}{}_{p}$ is zero unless $j = p$, in which event it is unity (why?),

$$\Gamma_{\alpha\beta\ p} a^{\alpha}{}_{r} a^{\beta}{}_{s} + m_{p\alpha} a^{\alpha}{}_{rs} = 0$$

Finally, on multiplying this relation by m^{jp} and summing with respect to p, where m^{jp} is the element in the jth row and pth column of M^{-1}, we obtain

$$m^{j\sigma} \Gamma_{\alpha\beta\ \sigma} a^{\alpha}{}_{r} a^{\beta}{}_{s} + a^{j}{}_{rs} = 0$$

We now differentiate the relation $\Gamma_{\alpha\beta\ p} a^{\alpha}{}_{r} a^{\beta}{}_{s} + m_{p\alpha} a^{\alpha}{}_{rs} = 0$ with respect to x^{t}, interchange s and t in the result, and subtract the two equations obtained in this way Since $a^{k}{}_{st} = a^{k}{}_{ts}$ and $a^{k}{}_{rst} = a^{k}{}_{rts}$ we obtain

$$(\Gamma_{\alpha\beta\ p})\,_{\gamma} a^{\alpha}{}_{r} a^{\beta}{}_{s} a^{\gamma}{}_{t} - (\Gamma_{\alpha\beta\ p})\,_{\gamma} a^{\alpha}{}_{r} a^{\beta}{}_{t} a^{\gamma}{}_{s} + \Gamma_{\alpha\beta,p} a^{\alpha}{}_{rt} a^{\beta}{}_{s}$$
$$- \Gamma_{\alpha\beta\ p} a^{\alpha}{}_{rs} a^{\beta}{}_{t} + m_{p\beta\ \gamma} a^{\beta}{}_{rs} a^{\gamma}{}_{t} - m_{p\beta\ \gamma} a^{\beta}{}_{rt} a^{\gamma}{}_{s} = 0$$

We simplify this relation by first writing the triple summation

$$(\Gamma_{\alpha\beta\ p})\,_{\gamma} a^{\alpha}{}_{r} a^{\beta}{}_{t} a^{\gamma}{}_{s}$$

in the equivalent form $(\Gamma_{\alpha\gamma\ p})\,_{\beta} a^{\alpha}{}_{r} a^{\beta}{}_{s} a^{\gamma}{}_{t}$ (why are these two triple summations the same?) and then eliminating the second derivatives of the a's with respect to the x's by means of the relation $a^{j}{}_{rs} = -m^{j\sigma} \Gamma_{\alpha\beta\ \sigma} a^{\alpha}{}_{r} a^{\beta}{}_{s}$ On the appropriate rearrangement of the summation labels every term has $a^{\alpha}{}_{r} a^{\beta}{}_{s} a^{\gamma}{}_{t}$ as a factor, and then if we multiply through by $x^{r}{}_{q} x^{s}{}_{j} x^{t}{}_{k}$ and sum on r, s, and t there remain only the terms for which $\alpha = q$, $\beta = j$ and $\gamma = k$ We obtain in this way the following relation

$$(\Gamma_{qj\ p})\,_{k} - (\Gamma_{qk\ p})\,_{j} - m^{\sigma\theta} \Gamma_{\sigma j\ p} \Gamma_{qk\ \theta} + m^{\sigma\theta} \Gamma_{qj\ \theta} \Gamma_{\sigma k\ p}$$
$$- m^{\sigma\theta} m_{p\sigma\ k} \Gamma_{qj\ \theta} + m^{\sigma\theta} m_{p\sigma\ j} \Gamma_{qk\ \theta} = 0$$

From the definition of the symbols $\Gamma_{pq\ r}$ it readily follows that $m_{pq\ r} =$

$\Gamma_{pr;q} + \Gamma_{qr;p}$, and, on using this to eliminate the first derivatives with respect to the a's of the elements of the matrix M, we obtain

$$R_{qp;jk} = (\Gamma_{qj;p})_{,k} - (\Gamma_{qk;p})_{,j} + m^{\epsilon\theta}\Gamma_{pj;\epsilon}\Gamma_{qk;\theta} - m^{\epsilon\theta}\Gamma_{pk;\epsilon}\Gamma_{qj;\theta} = 0.$$

These are the (exact) compatibility relations connecting the elements of M and their first and second derivatives with respect to the a's. The terms involving the second derivatives of the elements of M are furnished by $(\Gamma_{qj;p})_{,k} - (\Gamma_{qk;p})_{,j}$ and are $\frac{1}{2}(m_{jp,qk} - m_{jq,pk} - m_{pk,jq} + m_{qk,jp})$. It follows readily that the *curvature tensor* $R_{qp;jk}$ is alternating in p and q and that it is symmetric in the pairs qp and jk. There are, then, only six distinct compatibility relations, namely,

$$R_{23;23} = 0, \qquad R_{31;31} = 0, \qquad R_{12;12} = 0,$$

$$R_{23;31} = 0, \qquad R_{31;12} = 0. \qquad R_{12;23} = 0.$$

When the strain matrix is treated as an infinitesimal matrix, the symbols $\Gamma_{pq;r}$ are infinitesimals of at least the first order and so the terms involving the products of Γ's in the compatibility relations are of at least the second order. Neglecting second and higher order infinitesimals the compatibility relations reduce to

$$m_{jp,qk} - m_{jq,pk} - m_{pk,jq} + m_{qk,jp} = 0$$

or, equivalently, since M differs from 2η by a constant matrix,

$$\eta_{jp,qk} + \eta_{qk,jp} = \eta_{jq,pk} + \eta_{pk,jq}.$$

There are three relations of the type

$$\eta_{23,11} + \eta_{11,23} = \eta_{12,31} + \eta_{31,12}$$

and three of the type

$$\eta_{22,33} + \eta_{33,22} = 2\eta_{23,23}.$$

In our original notation, where we used (a, b, c) instead of (a^1, a^2, a^3), the six compatibility relations (of the infinitesimal theory of elasticity), are

$$\eta_{bc,aa} + \eta_{aa,bc} = \eta_{ab,ca} + \eta_{ca,ab},$$

$$\eta_{ca,bb} + \eta_{bb,ca} = \eta_{bc,ab} + \eta_{ab,bc},$$

$$\eta_{ab,cc} + \eta_{cc,ab} = \eta_{ca,bc} + \eta_{bc,ca},$$

$$\eta_{bb,cc} + \eta_{cc,bb} = 2\eta_{bc,bc},$$

$$\eta_{cc,aa} + \eta_{aa,cc} = 2\eta_{ca,ca},$$

$$\eta_{aa,bb} + \eta_{bb,aa} = 2\eta_{ab,ab}.$$

EXERCISES

1. Verify the (infinitesimal theory) compatibility relations by setting

$$\eta_{aa} = U_a, \ \eta_{bc} = \tfrac{1}{2}(W_b + V_c), \text{ etc}$$

2 Show that the (infinitesimal theory) compatibility relations are satisfied by every *homogeneous* strain, i e , by every strain matrix whose elements are constants Is this statement valid for the exact compatibility relations?

Answer Yes

3 Show that the (infinitesimal theory) compatibility relations are satisfied by any strain matrix whose elements are linear functions of (a, b, c) Is this statement true for the exact compatibility relations?

Answer No

3

THE CONNECTION BETWEEN STRESS AND STRAIN

1. The stress matrix

We consider any three-dimensional portion V_x of our deformable medium (when in the final or deformed state) and we denote the surface of V_x by S_x. Since our deformable medium is supposed to be in equilibrium, when in the deformed state, V_x is in equilibrium under the action of various forces. These forces are of two types:

(1) Mass or body forces (such as the weight of the material in V_x). ρ_x being the density at a typical point P_x:(x, y, z) of V_x, the mass of the element of volume dV_x is $\rho_x \, dV_x$ and we denote the force per unit mass acting on this element of mass by **F**; the coordinates of **F** are furnished by the elements of a 3×1 matrix

the elements of the 3×1 matrix

$$f = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

and the force acting on V_x across the element of area dS^x has its coordinates furnished by the elements of the 3×1 matrix $f\,dS_x$ where $dS_x = \{(dS^x)^* \, dS^x\}^{1/2}$ is the scalar element of area of S_x. We term the vector whose coordinates are furnished by the elements of the 3×1 matrix f the *stress* on the matrix element of area dS^x. Thus each coordinate of the stress vector has the dimensions of a force divided by an area. *Stress is force per unit area.*

We now make the basic assumption that these two systems of forces which maintain that portion of the *deformable* medium which occupies V_x in equilibrium would maintain it in equilibrium if it were *rigidified*. This implies that the two systems of forces (mass forces and surface forces) which act on the deformable material occupying V_x must satisfy, when taken together, the conditions imposed upon a system of forces which maintains in equilibrium a rigid body. These conditions are most conveniently expressed as follows:

The virtual work of all the forces of the system in any virtual rigid displacement is zero.

A virtual rigid displacement is a particular case of the more general concept of a *virtual deformation*. In order to define the latter we must imagine that the coordinates (x, y, z) of P_x are functions not only of the coordinates (a, b, c) of P_a but also of an accessory independent variable, or parameter, which we shall denote by θ. The four variables a, b, c and θ are independent, and if f is any differentiable function of these four variables we write its differential in the form $df + \delta f$ where

$$df = f_a \, da + f_b \, db + f_c \, dc, \qquad \delta f = f_\theta \, d\theta$$

Thus df is calculated under the assumption that θ is held constant whereas δf is calculated under the assumption that a, b, and c are held constant. A *virtual deformation* is defined when the 3×1 matrix

$$\delta x = \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

is given as a function of x, y, and z, i.e., when δx is furnished at each point P_x of the deformed position of the medium. When the 3×1 matrix δx is such that $\delta(ds_x)^2 = 0$ we say that the virtual deformation is a *virtual rigid displacement*.

Since the variables a, b, c and θ are independent the order in which second-order derivatives with respect to these variables are calculated is immaterial (these second-order derivatives being assumed continuous). Thus, for example, $x_{,a\theta} = x_{,\theta a}$ (where $x_{,a\theta}$ denotes the derivative with respect to θ of x_a, and $x_{,\theta a}$ denotes the derivative with respect to a of x_θ) and, after we multiply through by $d\theta$, this relation appears in the form $\delta(x_a) = (\delta x)_a$. If M is any matrix, of which the element in the qth row and pth column is $m_p{}^q$, we denote by δM the matrix of which the element in the qth row and pth column is $\delta m_p{}^q$. If we denote by $(\delta x)_x$ the 3×3 matrix

$$(\delta x)_x = \begin{pmatrix} (\delta x)_x & (\delta x)_y & (\delta x)_z \\ (\delta y)_x & (\delta y)_y & (\delta y)_z \\ (\delta z)_x & (\delta z)_y & (\delta z)_z \end{pmatrix} = \frac{(\delta x,\ \delta y,\ \delta z)}{(x,\ y,\ z)},$$

the relation $\delta(x_a) = (\delta x)_a$ and the similar relations obtained by replacing a by b or c and x by y or z yield

$$\delta J = (\delta x)_x J$$

where J is the Jacobian matrix $\dfrac{(x, y, z)}{(a, b, c)}$. [Prove this. *Hint.* $(\delta x)_a = (\delta x)_x x_a + (\delta x)_y y_a + (\delta x)_z z_a$.] It is clear that $\delta J^* = (\delta J)^*$ (prove this), and so

$$\delta J^* = J^*[(\delta x)_x]^*.$$

Since $M = J^*J$ it follows that

$$\begin{aligned} \delta M &= (\delta J^*)J + J^*\delta J \\ &= J^*\{[(\delta x)_x]^* + (\delta x)_x\}J. \end{aligned}$$

We denote by D the symmetric matrix

$$\begin{aligned} D &= \tfrac{1}{2}\{[(\delta x)_x]^* + (\delta x)_x\} \\ &= \begin{pmatrix} (\delta x)_x & \tfrac{1}{2}\{(\delta y)_x + (\delta x)_y\} & \tfrac{1}{2}\{(\delta x)_z + (\delta z)_x\} \\ \tfrac{1}{2}\{(\delta y)_x + (\delta x)_y\} & (\delta y)_y & \tfrac{1}{2}\{(\delta z)_y + (\delta y)_z\} \\ \tfrac{1}{2}\{(\delta x)_z + (\delta z)_x\} & \tfrac{1}{2}\{(\delta z)_y + (\delta y)_z\} & (\delta z)_z \end{pmatrix}, \end{aligned}$$

and then we have the relation

$$\delta M = 2J^*DJ,$$

or, equivalently, since $M = E_3 + 2\eta$

$$\delta\eta = J^*DJ.$$

Since $(ds_x)^2 = (da)^*M\,da$, we have [remember that a is independent of θ so that $\delta(da) = 0$]

$$\delta(ds_x)^2 = (da)^*\delta M\,da = 2(da)^*J^*DJ\,da$$
$$= 2(dx)^*D\,dx$$

For the virtual deformation to be a virtual rigid displacement we must have $\delta(ds_x)^2 = 0$ for arbitrary choice of the 3×1 matrix dx Hence we have the following criterion for a virtual rigid displacement

A virtual deformation δx is a virtual rigid displacement if, and only if, the symmetric matrix $D = \frac{1}{2}\{[(\delta x)_x]^* + (\delta x)_x\}$ is the zero matrix

It follows that the system of forces, both mass and surface, acting on V_x must be such that their virtual work, in any virtual deformation for which D is the zero matrix, is zero The simplest virtual rigid displacements are the *virtual translations* for which the elements of the 3×1 matrix δx are constant functions of (x, y, z) Setting, in particular $\delta x = 1$, $\delta y = 0$, $\delta z = 0$, we find that the virtual work of the mass forces is $\int_{V_x} \rho_x F\,dV_x$ whereas the virtual work of the surface forces is $\int_{S_x} f\,dS_x$ Hence

$$\int_{V_x} \rho_x F\,dV_x + \int_{S_x} f\,dS_x = 0$$

(which merely expresses the fact that the resolved parts in the direction of the positive x axis of all the forces, both mass and surface, acting on V_x add up to zero) On letting the volume of $V_x \to 0$ we see that $\lim\,(V_x \to 0) \int_{S_x} f\,dS_x = 0$, and on applying this result to a flat cylindrical volume whose height tends to zero we see that f changes sign when the direction of dS^x is reversed Applying our result to the tetrahedron whose faces are the coordinate planes and a plane whose coefficient vector is $v(u^1\ u^2, u^3)$, we find that

$$f = f_x u^1 + f_y u^2 + f_z u^3$$

where f_x is the x component of the stress on an element of area whose normal has the direction of the positive x-axis, etc Writing down the corresponding equations for g and h, we see that the 3×1 matrix f whose elements furnish the coordinates of the stress across the matrix element of area dS^x which has the direction of the unit vector $v(u^1, u^2, u^3)$ is given by the formula

$$f = Tu$$

the other hand, the stress matrix has nothing to do with the initial position of the medium, it is determined by the final position of the medium. Thus it does not make sense to ask: Is the stress matrix sensitive to a rotation of the initial rectangular Cartesian reference frame? It is logical to inquire, however: Is the stress matrix sensitive to a rotation of the final rectangular Cartesian reference frame?

EXERCISES

3. Show that if the stress on an element of area is always normal to the element the stress matrix is scalar, i.e., a multiple of E_3. *Note.* When $T = -pE_3$, $p > 0$, the stress on an element of area is always normal to the element and is a *pressure* rather than a *tension*. This is the situation in hydrostatics, and we term for this reason a scalar stress a *hydrostatic stress*. A hydrostatic stress is a *pressure* when T is a negative multiple of E_3 and a *tension* when T is a positive multiple of E_3.

4. Show that in a hydrostatic stress the magnitude of the stress on an element of area is independent of the direction of the element of area.

5. Show that the resolved part in the direction of the element of area dS^x of the stress on dS^x is u^*Tu, where u is the 3×1 matrix whose elements are the coordinates of the unit vector which furnishes the direction of dS^x.

2. The conditions of equilibrium and the virtual work in any virtual deformation

The virtual work of the mass forces acting on any portion V_x of the deformable medium in any virtual deformation is furnished by the volume integral $\int_{V_x} \rho_x(\delta x)^*F\,dV_x$ and the virtual work of the surface forces acting on V_x in this virtual deformation is furnished by the surface integral $\int_{S_x} (\delta x)^*f\,dS_x$. Since $f = Tu$ and since $(dS_x)u = dS^x$ the virtual work of the surface forces may be written in the form $\int_{S_x} (\delta x)^*T\,dS^x$. If, now, $\xi = (\xi, \eta, \zeta)$ is any 1×3 matrix the surface integral $\int_{S_x} \xi\,dS^x = \int_{S_x} (\xi\,dS^x + \eta\,dS^y + \zeta\,dS^z) = \int_{S_x} (\xi\,dy\,dz + \eta\,dz\,dx + \zeta\,dx\,dy)$ may be written, by virtue of Green's theorem, as the volume integral $\int_{V_x} (\xi_x + \eta_y + \zeta_z)\,dV_x = \int_{V_x} (\mathrm{div}_x\,\xi^*)\,dV_x$ where we understand by the divergence (in the x-coordinate system) of any $3 \times m$ matrix the $1 \times m$ matrix obtained by taking the divergence of each of its column vectors. Thus

$$\mathrm{div}_x\,\xi^* = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = \xi_x + \eta_y + \zeta_z$$

EXERCISE

1. Show that when the stress is a hydrostatic pressure the equations of equilibrium are $-(\text{grad}_x\, p)^* + \rho_x \Gamma = 0$ and that the equations of motion are $-(\text{grad}_x\, p)^* + \rho_x \Gamma = \rho_x x_{tt}$ where $\text{grad}_x\, p$ is the 1×3 matrix (p_x, p_y, p_z).

On using the fact that $T^* = T$, i.e., that the stress matrix is symmetric, in the expression for the virtual work, we obtain (show this) the following fundamental result:

The virtual work of all the forces acting on any portion V_x of our deformable medium in any virtual deformation is obtained by integrating the trace of the product TD over V_x:

$$\textit{Virtual work} = \int_{V_x} Tr(TD)\, dV_x.$$

We have seen that $\delta\eta = J^*DJ$ and so $TD = T(J^*)^{-1}\delta\eta J^{-1}$. Since $Tr\, AB = Tr\, BA$ it follows (show this) that

$$Tr(TD) = Tr(J^{-1}T(J^*)^{-1}\delta\eta).$$

Hence our expression for the virtual work of all the forces acting on any portion V_x of the deformable medium in any virtual deformation may be written in the equivalent form

Note If $a = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}$ and $b = \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}$ are two real 3×1 matrices, the scalar product of the column vector of b by the column vector of a is the 1×1 matrix a^*b This scalar product may be written as $Tr\, a^*b$ or, equivalently, as $Tr\, b^*a$ If A and B are two real 3×3 matrices, either of the two equal numbers $Tr\, A^*B$, $Tr\, B^*A$ may be termed, by analogy, the *scalar product* of the two 3×3 matrices A and B Thus the virtual work, in any virtual deformation, of all the forces acting on any portion V_x of our deformable medium is found by integrating the scalar product of $J^{-1}T(J^*)^{-1}$ and $\delta\eta$ over V_x It is clear that this *scalar* product is (as the name implies) insensitive to a rotation of the final rectangular Cartesian reference frame In fact, under the rotation $x \to x' = R^*x$, $J \to J' = R^*J$, $T \to T' = R^*TR$ and so $J^{-1}T(J^*)^{-1} \to J^{-1}RR^*TRR^*(J^*)^{-1} = J^{-1}T(J^*)^{-1}$ thus $J^{-1}T(J^*)^{-1}$ is insensitive to a rotation of the final rectangular Cartesian reference frame and we know that η, and hence $\delta\eta$, is insensitive to any such rotation In the infinitesimal theory of elasticity the (approximate) result concerning virtual work may be phrased as follows The virtual work of all the forces acting on any portion V_x of the deformable medium in any virtual displacement is found by integrating the scalar product of T and $\delta\eta$ over V_x Since T is sensitive to a rotation of the final rectangular Cartesian reference frame, whereas $\delta\eta$ is not, it is clear that the *scalar* product $Tr(T\delta\eta)$ of T and $\delta\eta$ depends on this final reference frame, this makes it evident that the statement of the infinitesimal theory of elasticity concerning virtual work cannot be an exact statement, for the virtual work cannot depend on the accidental choice of the final rectangular Cartesian reference frame

3. The virtual work when the stress is hydrostatic

The expression for the virtual work of all the forces acting on any portion V_x of the deformable medium, in any virtual deformation, takes a particularly simple form when the stress is hydrostatic, i e, when $T = -pE_3$ In this case $J^{-1}T(J^*)^{-1} = -pJ^{-1}(J^*)^{-1} = -p(M)^{-1}$ and so the virtual work is found by integrating the scalar product of $-p(M)^{-1}$ and $\delta\eta$ over V_x Since $M = E_3 + 2\eta$ $\delta M = 2\delta\eta$ and so

$$\text{Virtual work} = -\tfrac{1}{2}\int_{V_x} p\, Tr(M^{-1}\delta M)\, dV_x$$

If, now, A is any 3×3 matrix, $\delta(\det A)$ is the sum of three determinants (each obtained by differentiating *one* of the column vectors of A)

$$\delta(\det A) = \begin{vmatrix} \delta a_1{}^1 & a_2{}^1 & a_3{}^1 \\ \delta a_1{}^2 & a_2{}^2 & a_3{}^2 \\ \delta a_1{}^3 & a_2{}^3 & a_3{}^3 \end{vmatrix} + \begin{vmatrix} a_1{}^1 & \delta a_2{}^1 & a_3{}^1 \\ a_1{}^2 & \delta a_2{}^2 & a_3{}^2 \\ a_1{}^3 & \delta a_2{}^3 & a_3{}^3 \end{vmatrix} + \begin{vmatrix} a_1{}^1 & a_2{}^1 & \delta a_3{}^1 \\ a_1{}^2 & a_2{}^2 & \delta a_3{}^2 \\ a_1{}^3 & a_2{}^3 & \delta a_3{}^3 \end{vmatrix}$$

(prove this), and this result may be written in the equivalent form

$$\delta(\det A) = Tr(\text{co } A)^* \delta A.$$

If A is non-singular we obtain, on dividing through by det A,

$$Tr(A^{-1}\delta A) = (\det A)^{-1}\delta(\det A)$$

and so the virtual work may, when the stress is hydrostatic, be written in the form

$$\text{Virtual work} = -\tfrac{1}{2}\int_{V_x} p(\det M)^{-1}\delta(\det M)\, dV_x.$$

Since $\det M = (\det J)^2$ we have

$$\text{Virtual work} = -\int_{V_x} p(\det J)^{-1}\delta(\det J)\, dV_x,$$

and since $\det J = \dfrac{dV_x}{dV_a}$, so that $\delta(\det J) = \dfrac{1}{dV_a}\,\delta(dV_x)$, we obtain, finally,

$$\text{Virtual work} = -\int_{V_x} p\delta(dV_x).$$

If p is a constant function of (x, y, z), i.e., if the stress is a constant hydrostatic pressure (or tension), this equation simplifies further. On writing $V_x = \int_{V_x} dV_x = \int_{V_a} \dfrac{dV_x}{dV_a} dV_a$, we obtain $\delta V_x = \int_{V_a} \dfrac{\delta(dV_x)}{dV_a}\, dV_a$ $= \int_{V_x} \delta(dV_x)$. (Note that, in the integration over V_a, the limits of integration are independent of the accessory variable, or parameter, θ, whereas in the integration over V_x the limits of integration vary, in general, with θ.) Hence, when the stress is a constant hydrostatic pressure, or tension, we have

$$\text{Virtual work} = -p\int_{V_x} \delta(dV_x) = -p\delta V_x.$$

4. The elastic energy and the relation between stress and strain

We assume that the work done by all the forces acting on any portion V_x of our deformable medium in any deformation is stored up in V_x as *elastic energy* or *energy of deformation* (none of it being dissipated in heat). The energy of deformation may well depend not only on the

actual state of strain of the medium but also on the previous history of the medium, in other words, the energy of deformation may be distributed throughout V_x with a mass density ψ

$$\text{Energy of deformation} = \int_{V_x} \rho_x \psi \, dV_x$$

where ψ is a function not only of η but also of the previous values of η (from 0 to η). Such *hysteresis effects* are matters of common experience but we shall ignore them here. We make the fundamental assumption that the mass density ψ of the energy of deformation (i.e., the energy of deformation per unit mass) is a function of the strain matrix η so that ψ is unambiguously determinate when η is given

$$\psi = \psi(\eta)$$

Remark 1. It is more convenient to deal with the energy-per-unit mass rather than the energy-per unit-volume since the mass of any element of volume dV_x of the medium remains constant in any deformation whereas dV_x changes, in general.

Remark 2. Since η is a symmetric matrix, $\psi(\eta)$ is unambiguously determinate when the six following elements of η are given—η_{aa}, η_{bb}, η_{cc}, η_{bc}, η_{ca}, η_{ab}—and we can write

$$\psi = \psi(\eta_{aa}, \eta_{bb}, \eta_{cc}, \eta_{bc}, \eta_{ca}, \eta_{ab})$$

However, the gain obtained in reducing the number of variables of which ψ is a function from nine to six is overbalanced by the loss of symmetry involved in this reduction. We shall agree to write any function of the symmetric matrix η *symmetrically* where we understand that $f(\eta)$ is written symmetrically if $f(A) = f(A^*)$ where A is any 3×3 matrix. If $f(\eta)$ is any function of η it may be written symmetrically by setting $f(\eta) = f\left(\dfrac{\eta^* + \eta}{2}\right) = g(\eta)$, for it is obvious that $g(A) = g(A^*)$ where A is any 3×3 matrix (prove this). The main advantage of writing a function $f(\eta)$ of η symmetrically is that we do not lose symmetry when we take the *gradient* of f with respect to η. Just as the gradient of any function of (x, y, z) is the 1×3 matrix obtained by taking the three partial derivatives of the function with respect to x, y and z, so the gradient of any function $f(A)$ of a 3×3 matrix A is the 3×3 matrix obtained by taking the partial derivatives of $f(A)$ with respect to the various elements of A. Thus the element in the pth row and qth column of the gradient of $f(A)$ with respect to A is the partial derivative of $f(A)$ with respect to $a_p{}^q$. In calculating these partial derivatives we pay no attention to the particular structure

of A; we regard the nine elements of A as independent whether they actually are independent or not. Thus, in particular, if $A^* = A$ we regard a_q^p as constant when we are differentiating with respect to a_p^q even though we know that $a_q^p = a_p^q$. We denote the gradient of $f(A)$ by the symbol $\dfrac{\partial f}{\partial A}$. It is clear that $\dfrac{\partial f}{\partial A^*} = \left(\dfrac{\partial f}{\partial A}\right)^*$ and so we have the following important result:

If $f(A)$ is a function, written symmetrically, of the symmetric matrix A then $\dfrac{\partial f}{\partial A}$ is, like A, a symmetric matrix.

Example 1. The first invariant $I_1 = \eta_{aa} + \eta_{bb} + \eta_{cc}$ is a function, written symmetrically, of the symmetric matrix η. $\dfrac{\partial I_1}{\partial \eta}$ is the symmetric matrix E_3:

$$\frac{\partial I_1}{\partial \eta} = E_3.$$

Example 2. The second invariant $I_2 = (\eta_{bb}\eta_{cc} - \eta_{bc}\eta_{cb}) + (\eta_{cc}\eta_{aa} - \eta_{ca}\eta_{ac}) + (\eta_{aa}\eta_{bb} - \eta_{ab}\eta_{ba})$ is a function, written symmetrically, of the symmetric matrix η; $\dfrac{\partial I_2}{\partial \eta}$ is the symmetric matrix $I_1E_3 - \eta$:

$$\frac{\partial I_2}{\partial \eta} = I_1E_3 - \eta.$$

Example 3. The third invariant $I_3 = \det \eta$ is a function, (supposedly written symmetrically), of the symmetric matrix η. $\dfrac{\partial I_3}{\partial \eta}$ is the symmetric matrix co η:

$$\frac{\partial I_3}{\partial \eta} = \text{co } \eta.$$

The principle of conservation of energy may be expressed as follows: The virtual work of all the forces acting on any volume V_x of our deformable medium, in any virtual deformation, $= \delta \int_{V_x} \rho_x \psi \, dV_x$. On writing $\int_{V_x} \rho_x \psi \, dV_x$ in the equivalent form $\int_{V_a} \rho_a \psi \, dV_a$, we have $\delta \int_{V_x} \rho_x \psi \, dV_x = \int_{V_a} \rho_a \delta\psi \, dV_a = \int_{V_x} \rho_x \delta\psi \, dV_x$, and so the virtual work (i.e., the integral over V_x of $Tr(J^{-1}T(J^*)^{-1}\delta\eta)$) is the same as the

integral over V_x of $\rho_x \delta\psi$ Since this must hold for an arbitrary volume V_x of our deformable medium, we have (why?)

$$Tr(J^{-1}T(J^*)^{-1}\delta\eta) = \rho_x \delta\psi$$

Since ψ is (by hypothesis) a function (written symmetrically) of η, we have (by the very definition of a differential) $\delta\psi = Tr\left(\frac{\partial\psi}{\partial\eta}\,\delta\eta\right)$ (show this), and so

$$Tr(J^{-1}T(J^*)^{-1}\delta\eta) = \rho_x Tr\left(\frac{\partial\psi}{\partial\eta}\,\delta\eta\right)$$

Since this relation must hold for an arbitrary (symmetric) matrix $\delta\eta$, we have (why?)

$$J^{-1}T(J^*)^{-1} = \rho_x \frac{\partial\psi}{\partial\eta}$$

or, equivalently,

$$T = \rho_x J \frac{\partial\psi}{\partial\eta} J^*$$

On setting $\phi = \rho_a\psi$, so that ϕ is the energy of deformation *per unit initial volume* (why?), we obtain, finally,

$$T = \left(\frac{\rho_x}{\rho_a}\right) J \frac{\partial\phi}{\partial\eta} J^*$$

This is the fundamental relation of elasticity theory connecting stress and strain *If* ϕ is known as a function of η and *if* J is given, we can calculate the stress matrix T from this relation (η being determined by means of the formula $\eta = \frac{1}{2}(J^*J - E_3)$ and the compression ratio being determined by means of the formula $\frac{\rho_x}{\rho_a} = \frac{1}{\det J}$) In the infinitesimal theory of elasticity the difference between J and E_3 is neglected, (so that the difference between the compression ratio $\frac{\rho_x}{\rho_a}$ and 1 is neglected) and so $T = \frac{\partial\phi}{\partial\eta}$ In words,

To the degree of approximation afforded by the infinitesimal theory of elasticity, stress is the gradient of energy density with respect to strain, that is, the stress matrix is the gradient, with respect to the strain matrix, of the energy of deformation per unit initial volume

Be sure to understand that this is not an exact statement, it is only

an approximation to the exact result, which is furnished by the formula
$T = \left(\frac{\rho_x}{\rho_a}\right) J \frac{\partial\phi}{\partial\eta} J^*.$

Since $J^*J = M$ we have $J = RM^{1/2}$ where R is a rotation matrix; hence the formula furnishing T may be written as follows:

$$T = \frac{\rho_x}{\rho_a} RM^{1/2} \frac{\partial\phi}{\partial\eta} M^{1/2} R^*.$$

On comparing this with the relation

$$JJ^* = RMR^*,$$

we obtain the following result:

In that particular final rectangular Cartesian reference frame in which the coordinates of JJ^* are furnished by the elements of M the coordinates of T are furnished by the elements of $\left(\frac{\rho_x}{\rho_a}\right) M^{1/2} \frac{\partial\phi}{\partial\eta} M^{1/2}$.

If, in particular, the initial rectangular Cartesian reference frame is so chosen that M is in diagonal form, we see that:

In the final rectangular Cartesian reference frame that is furnished by the principal axes of JJ^* the coordinates of T are furnished by the elements of

$$\left(\frac{\rho_x}{\rho_a}\right)\begin{pmatrix} m_1^{1/2} & 0 & 0 \\ 0 & m_2^{1/2} & 0 \\ 0 & 0 & m_3^{1/2}\end{pmatrix} \frac{\partial\phi}{\partial\eta} \begin{pmatrix} m_1^{1/2} & 0 & 0 \\ 0 & m_2^{1/2} & 0 \\ 0 & 0 & m_3^{1/2}\end{pmatrix}$$

$$= \left(\frac{\rho_x}{\rho_a}\right)\begin{pmatrix} m_1 \frac{\partial\phi}{\partial\eta_{aa}} & (m_1m_2)^{1/2} \frac{\partial\phi}{\partial\eta_{ab}} & (m_1m_3)^{1/2} \frac{\partial\phi}{\partial\eta_{ac}} \\ (m_2m_1)^{1/2} \frac{\partial\phi}{\partial\eta_{ba}} & m_2 \frac{\partial\phi}{\partial\eta_{bb}} & (m_2m_3)^{1/2} \frac{\partial\phi}{\partial\eta_{bc}} \\ (m_3m_1)^{1/2} \frac{\partial\phi}{\partial\eta_{ca}} & (m_3m_2)^{1/2} \frac{\partial\phi}{\partial\eta_{cb}} & m_3 \frac{\partial\phi}{\partial\eta_{cc}} \end{pmatrix}$$

of unit magnitude and directed away from the medium) Since $dS_x u = dS^x$ we have

$$T\, dS^x = dS_x f$$

Since $T = \left(\frac{\rho_x}{\rho_a}\right) J \frac{\partial \phi}{\partial \eta} J^*$ and

$$dS^x = (\det J)\, (J^*)^{-1}\, dS^a = \left(\frac{\rho_a}{\rho_x}\right) (J^*)^{-1}\, dS^a$$

(See Chapter 1 Section 3) this relation may be written in the form

$$J \frac{\partial \phi}{\partial \eta}\, dS^a = dS_x f$$

We denote by T_a the 3×3 matrix $J \dfrac{\partial \phi}{\partial \eta}$ and observe that $T_a\, dS^a = T\, dS^x$, in words, *the result of operating on the final matrix element of area dS^x with the stress matrix T is the same as the result of operating on the initial matrix element of area dS^a with the matrix T_a* Since, in a deformation problem, it is the *initial* matrix element of area dS^a which is given (the determination of the final matrix element of area dS^x being part of the *problem*) it is with the matrix T_a and not with the stress matrix T, that we must operate when we wish to satisfy the *boundary conditions* These conditions are

$$T_a\, dS^a = dS_x f \qquad T_a = J \frac{\partial \phi}{\partial \eta}$$

If, for example, part of the surface of our deformable medium is free from applied force f is the zero matrix over this part of S_x and so $T_a\, dS^a = 0$ Since J is non singular it follows that $\dfrac{\partial \phi}{\partial \eta}\, dS^a = 0$ and so dS^a is a characteristic vector of $\dfrac{\partial \phi}{\partial \eta}$ corresponding to the characteristic number zero *Note* In some problems the stress matrix T (rather than the stress f) is furnished over S_x Thus the medium may be subjected to a constant hydrostatic pressure Then the boundary conditions simply state that T must coincide, over S_x, with the given stress matrix For example, an important problem that we shall later examine thoroughly is the following Determine the deformation of a circular cylindrical tube under the action of internal and external pressures p_i and p_e Assuming the displacement of each particle of the cylinder to be radial and a function of the distance of the particle

from the axis of the cylinder, the deformed tube is again a circular cylindrical tube. The vector $T_a \, dS^a$ must have the direction of the radius and, if r_i and r_e are, respectively, the *initial* internal and external radii of the tube, the magnitude of $T_a \, dS^a$ must be $p_i \, dS_x$ when $r = r_i$ and $p_e \, dS_x$ when $r = r_e$. From the relation $dS^x = \left(\frac{\rho_a}{\rho_x}\right) (J^*)^{-1} \, dS^a$, we obtain

$$dS_x = \{(dS^x)^* \, dS^x\}^{1/2} = \left(\frac{\rho_a}{\rho_x}\right) \{(dS^a)^* J^{-1} (J^*)^{-1} \, dS^a\}^{1/2}$$

$$= \left(\frac{\rho_a}{\rho_x}\right) \{(dS^a)^* M^{-1} \, dS^a\}^{1/2}.$$

4

ISOTROPIC ELASTIC MEDIA

1. The energy of deformation of an isotropic elastic medium

We have seen that the strain matrix η is sensitive to a rotation of the initial rectangular Cartesian reference frame Under the rotation $a \to a' = R^*a$, $\eta \to \eta' = R^*\eta R$ This change of the strain matrix does not mean that the rotation of the axes affects the strain or deformation, the strain is the same as before but its coordinates (i e , the elements of the strain matrix) with respect to the new axes are, in general, different from its coordinates with respect to the old axes The energy of deformation, per unit initial volume, must be insensitive to the rotation $a \to a' = R^*a$ of the initial rectangular Cartesian reference frame This fact implies, since η is sensitive, in general, to such a rotation, that the *form* of the function $\phi(\eta)$, which furnishes the energy per unit initial volume, must be, in general, sensitive to a rotation of the initial rectangular Cartesian reference frame In the new reference frame the energy of deformation, per unit initial volume, will be furnished by a *new* function ϕ' of the new strain matrix η', and the fact that the energy of deformation, per unit initial volume, is a *scalar* (i e , that it is insensitive to any rotation $a \to a' = R^*a$ of the initial rectangular Cartesian reference frame) assures us that

$$\phi(\eta) = \phi'(\eta') = \phi'(R^*\eta R)$$

If it should happen that the *form* of the function ϕ is insensitive to a given rotation $a \to a' = R^*a$ of the initial rectangular Cartesian reference frame, we say that the medium is *elastically insensitive* to the rotation R Thus the medium is elastically insensitive to a given rotation R if, and only if, $\phi'(R^*\eta R) = \phi(R^*\eta R)$ for an arbitrary symmetric matrix η In view of the relation $\phi(\eta) = \phi'(R^*\eta R)$ we can rephrase this definition of *elastic symmetry* (with respect to a given rotation of the initial rectangular Cartesian reference frame) as follows

The deformable medium is elastically insensitive to the rotation $a \to$

$a' = R^*a$ of the initial rectangular Cartesian reference frame if, and only if,

$$\phi(\eta) = \phi(R^*\eta R),$$

this relation being an identity in the (symmetric) matrix η.

Instead of regarding the elements of $\eta' = R^*\eta R$ as furnishing the coordinates of the strain with respect to the new axes we may consider the *new* strain whose coordinates with respect to the original axes are the same as the coordinates of the *old* strain with respect to the new axes (so that the coordinates of the new strain with respect to the original axes are furnished by the elements of $\eta' = R^*\eta R$). Then the relation $\phi(\eta) = \phi(R^*\eta R)$ may be interpreted as indicating that the energies of deformation, per unit volume, of these *two* (in general, different) strains are the same. The new strain is that which would be produced if the original strain were *preceded* by the rigid rotation $a \to a' = R^*a$ of the medium. Note that this is an actual rotation of the medium and not merely a rotation of the initial rectangular Cartesian reference frame.

A deformable medium is said to be *elastically isotropic* (or simply *isotropic*) if it is elastically insensitive to *every* rotation of the initial rectangular Cartesian reference frame. Thus,

A deformable medium is isotropic if, and only if,

isotropic medium $\dfrac{\partial\phi}{\partial\eta} = \dfrac{\partial\phi}{\partial I_1} E_3 + \dfrac{\partial\phi}{\partial I_2}(I_1E_3 - \eta) + \dfrac{\partial\phi}{\partial I_3}\,\text{co}\,\eta$, hence when the initial rectangular Cartesian reference frame is so chosen that η (and hence M and $M^{1/2}$) are diagonal so also is $\dfrac{\partial\phi}{\partial\eta}$ It follows that $\dfrac{\partial\phi}{\partial\eta}$ commutes with $M^{1/2}$ so that $M^{1/2}\dfrac{\partial\phi}{\partial\eta}M^{1/2} = M\dfrac{\partial\phi}{\partial\eta}$ Hence,

For an isotropic medium the coordinates of the stress tensor T are furnished in the final rectangular Cartesian reference frame in which the coordinates of JJ^* are furnished by the elements of $M = J^*J$, by the elements of the matrix $\left(\dfrac{\rho_x}{\rho_a}\right) M\dfrac{\partial\phi}{\partial\eta}$

In the final rectangular Cartesian reference frame that is furnished by the principal axes of JJ^* T is diagonal (it being understood that the axes of the initial rectangular Cartesian reference frame are the principal axes of $M = J^*J$)

$$T = \left(\frac{\rho_x}{\rho_a}\right)\begin{pmatrix} m_1\dfrac{\partial\phi}{\partial\eta_1} & 0 & 0 \\ 0 & m_2\dfrac{\partial\phi}{\partial\eta_2} & 0 \\ 0 & 0 & m_3\dfrac{\partial\phi}{\partial\eta_3}\end{pmatrix}$$

where $\dfrac{\partial\phi}{\partial\eta_1} = \dfrac{\partial\phi}{\partial I_1} + (\eta_2 + \eta_3)\dfrac{\partial\phi}{\partial I_2} + \eta_2\eta_3\dfrac{\partial\phi}{\partial I_3}$ etc Thus the principal axes of T are the principal axes of JJ^* In an arbitrary final rectangular Cartesian reference frame we have

$$R^*TR = \left(\frac{\rho_x}{\rho_a}\right)\begin{pmatrix} m_1\dfrac{\partial\phi}{\partial\eta_1} & 0 & 0 \\ 0 & m_2\dfrac{\partial\phi}{\partial\eta_2} & 0 \\ 0 & 0 & m_3\dfrac{\partial\phi}{\partial\eta_3}\end{pmatrix}$$

where $J = RM^{1/2}$ *Note* The principal axes of η are the principal axes of J^*J whereas the principal axes of T are for an isotropic medium the principal axes of JJ^* Thus the principal axes of T are not even for an isotropic medium in general the same as the principal axes of η If however we *follow* the deformation by the rigid rota

$\frac{l+2m}{3} I_1^3 - 2m\, I_1 I_2 + n I_3$ We shall not carry the development of ϕ, as a power series in I_1, I_2, I_3, farther (so that we agree to neglect, in the development of ϕ, infinitesimals of higher order than the third in the elements of the strain matrix η) We have, then,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3,$$

$$\phi_1 = \alpha I_1, \qquad \frac{\partial \phi_1}{\partial \eta} = E_3,$$

$$\phi_2 = \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2, \qquad \frac{\partial \phi_2}{\partial \eta} = \lambda I_1 E_3 + 2\mu\eta;$$

$$\phi_3 = \frac{l+2m}{3} I_1^3 - 2m I_1 I_2 + n I_3, \qquad \frac{\partial \phi_3}{\partial \eta} = (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n \text{ co } \eta,$$

$$\frac{\partial \phi}{\partial \eta} = \alpha E_3 + (\lambda I_1 E_3 + 2\mu\eta) + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n \text{ co } \eta$$

Hence $T = \left(\frac{\rho_x}{\rho_a}\right) J\{\alpha E_3 + (\lambda I_1 E_3 + 2\mu\eta) + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n \text{ co } \eta\}J^*$ When the strain is zero, η is the zero matrix and J is a rotation matrix On denoting by T_0 the *initial* stress matrix, i e, the stress matrix when $\eta = 0$, it follows (since $\frac{\rho_x}{\rho_a} = 1$ when $\eta = 0$ [why?]) that

$$T_0 = \alpha E_3$$

In other words, *the initial stress is either a hydrostatic pressure or a hydrostatic tension* The only assumption about the deformable medium which we have made is that the medium is isotropic and this has been enough to force the conclusion that the initial stress must be a scalar stress, i e, a hydrostatic pressure or tension Hence,

No elastic medium that is initially in a state of stress which is not scalar, i e, of the nature of a hydrostatic pressure or tension, can be isotropic

Assuming that the medium is isotropic, we denote the initial hydrostatic pressure by p_0, then $\alpha = -p_0$ and the stress matrix T, when the strain matrix is η, is furnished by the formula

$$T = \left(\frac{\rho_x}{\rho_a}\right) J\{-p_0 E_3 + (\lambda I_1 E_3 + 2\mu\eta) + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n \text{ co } \eta\}J^*$$

This formula involves the six *constants* p_0, λ, μ, l, m, n (the significance of the term constant being that they are independent of the strain matrix η). Of these six constants the first is the initial pressure and so its dependence on the initial pressure is obvious. It is natural to expect, then, that the remaining five, namely, λ, μ, l, m, and n, depend also on p_0. These five constants are known as the *elastic constants* of the isotropic medium. In the infinitesimal theory, which neglects, in the formula for the stress matrix, infinitesimals of the second order in the elements of η, the three elastic constants (l, m, n) do not appear, and T is furnished by the formula

$$T = \left(\frac{\rho_x}{\rho_a}\right) J\{-p_0E_3 + (\lambda I_1E_3 + 2\mu\eta)\}J^*$$

where the compression ratio $\dfrac{\rho_x}{\rho_a} = 1 - I_1$. The two elastic constants λ and μ which appear in this formula are known as *the elastic constants of Lamé* (after Gabriel Lamé [1795–1870], a French mathematician).

Warning. Do not be misled by the term *constant* into overlooking the fact that λ and μ depend on the initial hydrostatic pressure.

Writing $J = RM^{1/2}$, we have, to the degree of approximation contemplated by the infinitesimal theory,

$$\begin{aligned} R^*TR &= (1 - I_1)(E_3 + 2\eta)\{-p_0E_3 + (\lambda I_1E_3 + 2\mu\eta)\} \\ &= -p_0E_3 + (\lambda + p_0)I_1E_3 + 2(\mu - p_0)\eta. \end{aligned}$$

When the initial stress is zero, this equation reduces to

$$R^*TR = \lambda_0 I_1E_3 + 2\mu_0\eta$$

where (λ_0, μ_0) are the values of (λ, μ) that correspond to $p_0 = 0$, i.e., to zero initial pressure. On comparing this with the previous relation, which may be written in the form

$$R^*(T + p_0E_3)R = (\lambda + p_0)I_1E_3 + 2(\mu - p_0)\eta,$$

it is clear that the numbers λ_0 and μ_0 that appear in the formula furnishing the stress, when the medium is free from stress in the initial state from which the strain is measured, are the values of $\lambda + p_0$ and $\mu - p_0$, respectively, when p_0 is zero. If, for example, we neglect the dependence of λ and μ on p_0 the effect of a non-zero initial hydrostatic pressure is, in so far as the *increment* $T + p_0E_3$ in the stress matrix is concerned, to *increase* λ by p_0 and to *decrease* μ by p_0.

In the second-order approximation, in which we neglect, in the formula for T, third-order (but keep second-order) infinitesimals in

the elements of the strain matrix η, the formula for R^*TR is

$$R^*TR = \frac{\rho_x}{\rho_a}(E_3 + 2\eta)\{-p_0E_3 + (\lambda I_1E_3 + 2\mu\eta) + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n\,\mathrm{co}\,\eta\}$$

where now $\frac{\rho_x}{\rho_a} = (1 + 2I_1 + 4I_2)^{-1/2} = 1 - I_1 + \left(\frac{3}{2}I_1^2 - 2I_2\right)$ On collecting terms, we obtain $R^*TR = -p_0E_3 + (\lambda + p_0)I_1E_3 + 2(\mu - p_0)\eta + 2(\lambda + p_0 - \mu + m)I_1 + [(l - \lambda - \frac{3}{2}p_0)I_1^2 - 2(m - p_0)I_2]E_3 + 4\mu\eta^2 + n\,\mathrm{co}\,\eta$ When the medium is initially free from stress this reduces to $R^*TR = (\lambda_0I_1E_3 + 2\mu_0\eta) + 2(\lambda_0 - \mu_0 + m_0)I_1 + \{(l_0 - \lambda_0)I_1^2 - 2m_0I_2\}E_3 + 4\mu_0\eta^2 + n_0\,\mathrm{co}\,\eta$ where $(l_0\ m_0\ n_0)$ are the values of (l, m, n) that correspond to $p_0 = 0$

In comparisons of theory with experiment the compression ratio $\frac{\rho_x}{\rho_a}$ is often furnished by experiment and it is unnecessary to use the approximation ($1 - I_1$ in the linear theory and $1 - I_1 + (\frac{3}{2}I_1^2 - 2I_2)$ in the second order approximation) If we do not use these approximations we obtain the following results

Linear Theory

$$R^*TR = \left(\frac{\rho_x}{\rho_a}\right)(E_3 + 2\eta)\{-p_0E_3 + (\lambda I_1E_3 + 2\mu\eta)\}$$
$$= \left(\frac{\rho_x}{\rho_a}\right)\{-p_0E_3 + \lambda I_1E_3 + 2(\mu - p_0)\eta\}$$

When $p_0 = 0$ this reduces to

$$R^*TR = \left(\frac{\rho_x}{\rho_a}\right)(\lambda_0I_1E_3 + 2\mu_0\eta)$$

Second order Approximation

$$\begin{aligned} R^*TR &= \left(\frac{\rho_x}{\rho_a}\right)(E_3 + 2\eta)\{-p_0E_3 + (\lambda I_1E_3 + 2\mu\eta) \\ &\quad + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n\,\mathrm{co}\,\eta\} \\ &= \left(\frac{\rho_x}{\rho_a}\right)\{-p_0E_3 + \lambda I_1E_3 + 2(\mu - p_0)\eta \\ &\quad + (lI_1^2 - 2mI_2)E_3 + 2(m + \lambda)I_1\eta + n\,\mathrm{co}\,\eta + 4\mu\eta^2\} \end{aligned}$$

When $p_0 = 0$ this reduces to

$$R^*TR = \left(\frac{\rho_x}{\rho_a}\right)\{\lambda_0I_1E_3 + 2\mu_0\eta + (l_0I_1^2 - 2m_0I_2)E_3 + 2(m_0 + \lambda_0)I_1\eta + n_0\,\mathrm{co}\,\eta + 4\mu_0\eta^2\}$$

3. The stress in an isotropic medium when in a state of scalar strain

We shall write our scalar strain in the form

$$\eta = -cE_3$$

(the reason for the negative sign being that most of the experiments with which we wish to check the theory deal with *compression* rather than *dilatation*. If we wrote $\eta = cE_3$ the coefficient c would be positive in dilatation and negative in compression; when we write $\eta = -cE_3$ the coefficient c is positive in compression and negative in dilatation). Since the medium is isotropic, $\dfrac{\partial \phi}{\partial \eta}$ is a scalar matrix (i.e., a multiple of E_3) when η is a scalar matrix (prove this) and it follows (why?) that R^*TR is a scalar matrix. Hence (why?) T is a scalar matrix (being the same as R^*TR), and on setting

$$T = -pE_3$$

(so that p is the pressure in the medium) we obtain the following relations between c and p in the linear theory and the second-order approximation, respectively:

Linear Theory. Since $(1 - 2c) = \left(\dfrac{\rho_x}{\rho_a}\right)^{-2/3}$ the formula for p may be written as follows:

$$p = \left(\frac{\rho_x}{\rho_a}\right)^{1/3} \{p_0 + (3\lambda + 2\mu)c\}.$$

When we use the approximation $1 + 3c$ to $\dfrac{\rho_x}{\rho_a}$ this equation becomes

in other words, the compressibility is the derivative with respect to p of the logarithm of $\frac{1}{V}$ Terming the reciprocal of the compressibility the *incompressibility*, we have the following result

To the degree of approximation furnished by the infinitesimal (or linear) theory of elasticity the (local) incompressibility is given by the formula

$$\textit{Local incompressibility} = \lambda + \tfrac{2}{3}\mu + \tfrac{1}{3}p_0$$

When the initial pressure is zero the local incompressibility is $\lambda_0 + \frac{2}{3}\mu_0$

Note carefully that the (local) incompressibility is sensitive to the value of the initial pressure If we neglect the dependence of λ and μ on p_0 (so that we replace λ and μ by λ_0 and μ_0, respectively), the (local) incompressibility is the linear function $\lambda_0 + \frac{2}{3}\mu_0 + \frac{1}{3}p_0$ of p_0 If we make the less drastic assumption that λ and μ are linear functions of p_0, the (local) incompressibility is a linear function $\lambda_0 + \frac{2}{3}\mu_0 + kp_0$ of p_0 (the coefficient k of p_0 being a constant that must be determined by experiment) Assuming that the strain and p_0 are constant over the homogeneous medium, we have, then,

$$\left(\frac{d}{dp}\log\frac{1}{V}\right)_{p=p_0} = \frac{1}{\lambda_0 + \frac{2}{3}\mu_0 + kp_0}$$

Since p_0 is arbitrary, this equation is equivalent to the relation

$$\frac{d}{dp}\log\frac{1}{V} = \frac{1}{\lambda_0 + \frac{2}{3}\mu_0 + kp}$$

On integrating this relation, we obtain the following *equation of state* for the isotropic deformable medium

$$1 + \frac{k}{\lambda_0 + \frac{2}{3}\mu_0}\,p = \left(\frac{V_0}{V}\right)^k$$

or, equivalently,

$$p = \frac{(\lambda_0 + \frac{2}{3}\mu_0)}{k}\left\{\left(\frac{V_0}{V}\right)^k - 1\right\}$$

where V_0 is the value of V which corresponds to $p = 0$ The value of k which corresponds to the (drastic) assumption that λ and μ are independent of p_0 is $\frac{1}{3}$, and the corresponding equation of state is

$$p = (3\lambda + 2\mu)\left\{\left(\frac{V_0}{V}\right)^{\frac{1}{3}} - 1\right\}$$

The Second-order Approximation. Here

$$p = \left(\frac{\rho_x}{\rho_a}\right)^{\frac{1}{3}} \{p_0 + (3\lambda + 2\mu)e - (9l + n)e^2\},$$

and this reduces, when $p_0 = 0$, to

$$p = \left(\frac{\rho_x}{\rho_a}\right)^{\frac{1}{3}} \{(3\lambda_0 + 2\mu_0)e - (9l_0 + n_0)e^2\}.$$

Since $\left(\frac{\rho_x}{\rho_a}\right)^{\frac{1}{3}} = (1 - 2e)^{-\frac{1}{2}} = 1 + e + \frac{3}{2}e^2 + \cdots$, these results may be written in the equivalent forms

$$p = p_0 + (3\lambda + 2\mu + p_0)e + (3\lambda + 2\mu - 9l - n + \tfrac{3}{2}p_0)e^2,$$

$$p = (3\lambda_0 + 2\mu_0)e + (3\lambda_0 + 2\mu_0 - 9l_0 - n_0)e^2.$$

Limiting ourselves to the case in which p_0 and e are constant over the medium, we have $\frac{\rho_a}{\rho_x} = \frac{V_x}{V_a} = 1 + \frac{\Delta V}{V_0}$ and so $(1 - 2e)^{\frac{3}{2}} = 1 + \frac{\Delta V}{V_0}$ or, equivalently, $1 - 2e = \left(1 + \frac{\Delta V}{V_0}\right)^{\frac{2}{3}} = 1 + \frac{2}{3}\frac{\Delta V}{V_0} - \frac{1}{9}\left(\frac{\Delta V}{V_0}\right)^2 + \cdots$. Hence $e = -\frac{1}{3}\frac{\Delta V}{V_0} + \frac{1}{18}\left(\frac{\Delta V}{V_0}\right)^2 + \cdots$ and so the formulas for p may be written as follows:

$$\Delta p = -\left(\lambda + \frac{2}{3}\mu + \frac{1}{3}p_0\right)\frac{\Delta V}{V_0} + \frac{1}{2}\left(\lambda + \frac{2}{3}\mu - 2l - \frac{2}{9}n + \frac{4}{9}p_0\right)\left(\frac{\Delta V}{V_0}\right)^2,$$

$$p = -\left(\lambda_0 + \frac{2}{3}\mu_0\right)\frac{\Delta V}{V_0} + \frac{1}{2}\left(\lambda_0 + \frac{2}{3}\mu_0 - 2l_0 - \frac{2}{9}n_0\right)\left(\frac{\Delta V}{V_0}\right)^2.$$

If $\log \frac{1}{V}$ is denoted, for the moment, by ξ, it follows that

$$\frac{dp}{d\xi} = \left(\lambda + \frac{2}{3}\mu + \frac{1}{3}p\right), \qquad \frac{d^2p}{d\xi^2} = \lambda + \frac{2}{3}\mu - 2l - \frac{2}{9}n + \frac{4}{9}p.$$

On differentiating the first of these relations with respect to ξ and substituting the result in the second, we obtain

$$\frac{d}{d\xi}\left(\lambda + \frac{2}{3}\mu\right) = \frac{2}{3}\left(\lambda + \frac{2}{3}\mu\right) - 2l - \frac{2}{9}n + \frac{1}{3}p.$$

and if this value is divided by the expression for $\frac{dp}{d\xi}$ it follows that

$$\frac{d}{dp}\left(\lambda + \frac{2}{3}\mu\right) = \frac{\frac{2}{3}(\lambda + \frac{2}{3}\mu) - 2l - \frac{2}{9}n + \frac{1}{3}p}{\lambda + \frac{2}{3}\mu + \frac{1}{3}p}$$

For values of p which are negligible in comparison with $\lambda_0 + \frac{2}{3}\mu_0$ a good approximation to the fraction on the right is

$$\frac{2}{3}\left\{1 - \frac{9l_0 + n_0}{3\lambda_0 + 2\mu_0}\right\}$$

Thus the k that appears in the equation of state

$$p = \frac{(\lambda_0 + \frac{2}{3}\mu_0)}{k}\left\{\left(\frac{V_0}{V}\right)^k - 1\right\}$$

of the linear theory is (approximately) $1 - \dfrac{2(9l_0 + n_0)}{3(3\lambda_0 + 2\mu_0)}$ The fact that $\lambda_0 + \frac{2}{3}\mu_0$ may be determined by experiment (as the value of $\frac{dp}{d\xi}$ when $p = 0$) furnishes a method of determining by experiment the combination $9l_0 + n_0$

Remark 1 We have termed the equation

$$p = \frac{(\lambda_0 + \frac{2}{3}\mu_0)}{k}\left\{\left(\frac{V_0}{V}\right)^k - 1\right\}$$

the equation of state of the linear theory Actually, the equation of state of the linear theory is the equation

$$p = (3\lambda_0 + 2\mu_0)\left(\frac{\rho_x}{\rho_a}\right)^{1/3} e,$$

which we obtain by setting $p_0 = 0$ in the equation

$$p = \left(\frac{\rho_x}{\rho_a}\right)^{1/3} \{p_0 + (3\lambda + 2\mu)e\}$$

If we use the approximation $\frac{\rho_x}{\rho_a} = 1 + 3e$ this linear approximation appears in the form

$$p = (3\lambda_0 + 2\mu_0)e = \left(\lambda_0 + \frac{2}{3}\mu_0\right)\frac{\Delta\rho}{\rho_a} = -\left(\lambda_0 + \frac{2}{3}\mu_0\right)\frac{\Delta V}{V_0}$$

This equation is in disagreement with experimental results unless $\frac{\Delta V}{V_0}$ is infinitesimal, and we shall not stop to discuss it. The equation that we prefer to call the equation of state of the linear theory is really the equation of state of an *integrated linear theory.* We use the linear theory to express Δp as a multiple of $\frac{\Delta V}{V}$ not merely at $p = 0$ but also at *an arbitrary* value of p; the multiple $\lambda + \frac{2}{3}\mu + \frac{1}{3}p$ is an *unknown* function of p, and we approximate it by a linear function of p (the coefficient of p in this linear function being determined by experiment). The final relation between p and V is obtained by *integrating* the relation between dp and dV obtained in this way.

Remark 2. The formula $p = \left(\frac{\rho_x}{\rho_a}\right)^{1/3} \{p_0 + (3\lambda + 2\mu)c - (9l + n)c^2\}$ of the second-order approximation may be checked against the relation $p = -\frac{\partial \Phi}{\partial V_x}$ where $\Phi = V_a\phi$ is the energy of deformation of the entire medium. In fact, on setting $I_1 = -3c$, $I_2 = 3c^2$, $I_3 = -c^3$ in the formula

$$\phi = -p_0 I_1 + \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + \frac{l + 2m}{3} I_1^3 - 2m I_1 I_2 + n I_3,$$

we obtain

$$\phi = 3p_0 c + \tfrac{3}{2}(3\lambda + 2\mu)c^2 - (9l + n)c^3$$

and so

$$\Phi = V_a \{3p_0 c + \tfrac{3}{2}(3\lambda + 2\mu)c^2 - (9l + n)c^3\}.$$

Since $(1 - 2c)^3 = \left(\frac{V_x}{V_a}\right)^2$ we have $c = \frac{1}{2}\left\{1 - \left(\frac{V_x}{V_a}\right)^{2/3}\right\}$ and so $\frac{\partial c}{\partial V_x} = -\frac{1}{3}\left(\frac{V_x}{V_a}\right)^{-1/3} \frac{1}{V_a}$. Hence $-\frac{\partial \Phi}{\partial V_x} = \left(\frac{V_x}{V_a}\right)^{-1/3} \{p_0 + (3\lambda + 2\mu)c - (9l + n)c^2\} = \left(\frac{\rho_x}{\rho_a}\right)^{1/3} \{p_0 + (3\lambda + 2\mu)c - (9l + n)c^2\}$.

4. Comparison of theory with experiment in the case of extreme hydrostatic pressure

The experiments of Bridgman on the compressibilities of various media up to the extreme pressure of 10^5 atmospheres ($= 10^8$ grams per square centimeter) are available to test the various relations connecting pressure with volume which are furnished by the various theories we have considered (the simple linear theory, the integrated linear

theory and the second order approximation) Bridgman[1] gives the relative volume $\frac{V}{V_0}$ at pressures up to 10^5 atmospheres (1 atmosphere being 1 kilogram per square centimeter) The values for the metal sodium are as follows (the unit of pressure being taken as 10^4 atmospheres $= 10^7$ grams per square centimeter)

$p =$	2 5	3	4	5	6	7	8	9	10
$\frac{V}{V_0} = \left(1 + \frac{\Delta V}{V_0}\right) -$	0 789	0 770	0 737	0 708	0 683	0 661	0 641	0 623	0 606
$-\frac{\Delta V}{V_0} = 1 - \frac{V}{V_0} =$	0 211	0 230	0 263	0 292	0 317	0 339	0 359	0 377	0 394

It is immediately clear that the formula furnished by the (simple) linear theory is in disagreement with the experimental results This formula is

$$p = -\left(\lambda_0 + \frac{2}{3}\mu_0\right)\frac{\Delta V}{V_0},$$

and if we use the experimentally determined value of $\frac{\Delta V}{V_0}$ at $p = 2\ 5$ to determine the value of the (single) constant $\lambda_0 + \frac{2}{3}\mu_0$ appearing in the formula we find that

$$\lambda_0 + \tfrac{2}{3}\mu_0 = 11\ 85$$

The following table gives the comparison between the experimentally determined values of $\frac{V}{V_0}$ and the values of $\frac{V}{V_0}$ calculated from the formula

$$\frac{V}{V_0} = 1 + \frac{\Delta V}{V_0} = 1 - \frac{p}{\lambda_0 + \frac{2}{3}\mu_0} = 1 - 0\ 0844p$$

$p =$	2 5	3	4	5	6	7	8	9	10
$\frac{V}{V_0}$ (measured) $=$	0 789	0 770	0 737	0 708	0 683	0 661	0 641	0 623	0 606
$\frac{V}{V_0}$ (calculated) $=$	0 789	0 747	0 662	0 578	0 494	0 409	0 325	0 240	0 156

[1] *Proc Am Acad Arts Sci* Vol 76 pp 55–70 (1948)

When $p = 4$ the value of $\frac{V}{V_0}$ given by the formula is too small by about 10 per cent. The experimental values of $\frac{V}{V_0}$ have an accuracy of 2 or 3 per cent, and so the simple linear theory is not valid up to a pressure of 4.10^4 atmospheres (less than twice the pressure used in determining the constant $\lambda_0 + \frac{2}{3}\mu_0$ of the formula). We shall not consider further, therefore, the formula furnished by the simple linear theory and shall pass to a study of the formula provided by the integrated linear theory.

There are two constants, namely, k and $\lambda_0 + \frac{2}{3}\mu_0$ in the formula

$$p = \frac{(\lambda_0 + \frac{2}{3}\mu_0)}{k}\left\{\left(\frac{V_0}{V}\right)^k - 1\right\}$$

of the integrated linear theory. If p_1 and p_2 are two pressures to which correspond the volumes V_1 and V_2, respectively, we have

$$\frac{p_2}{p_1} = \frac{\left(\frac{V_0}{V_2}\right)^k - 1}{\left(\frac{V_0}{V_1}\right)^k - 1}$$

and so $p_2\left(\frac{V_0}{V_1}\right)^k - p_1\left(\frac{V_0}{V_2}\right)^k - (p_2 - p_1) = 0$. The value of k may be determined from this equation (by the method of trial and error) by using the experimental values of $\frac{V_0}{V_1}$ and $\frac{V_0}{V_2}$; k having been determined, the value of $\lambda_0 + \frac{2}{3}\mu_0$ may be obtained from the equation

$$\lambda_0 + \frac{2}{3}\mu_0 = \frac{kp_2}{\left(\frac{V_0}{V_2}\right)^k - 1}.$$

In order to see how the integrated linear theory works both backwards and forwards, i.e., when we extrapolate to lower as well as to higher values of p, we select the values $p_1 = 5$, $p_2 = 6$ in order to determine the two constants, k and $\lambda_0 + \frac{2}{3}\mu_0$, of the formula. Our calculation will serve to determine the value of the integrated linear theory as a means of *prediction;* thus we may imagine that Bridgman had succeeded in measuring $\frac{V}{V_0}$ only up to $p = 6$ (i.e., up to a pressure of

60,000 atmospheres) and we wish to predict the values of $\frac{V}{V_0}$ up to $p = 10$ (i.e., up to a pressure of 10^6 atmospheres). If our predicted values check well with the experimentally determined values we may use these values to predict the (as yet experimentally undetermined) values of $\frac{V}{V_0}$ up to $p = 20$ (i.e., up to a pressure of 200,000 atmospheres).

The equation that serves to determine k is

$$6\left(\frac{V_0}{V_1}\right)^k - 5\left(\frac{V_0}{V_2}\right)^k - 1 = 0$$

where $\frac{V_1}{V_0} = 0.708$, $\frac{V_2}{V_0} = 0.683$. On denoting the left-hand side of this equation by $f(k)$ we find that (check these calculations) $f(3) = 0.214$, $f(4) = -0.0965$, $f(3.8) = -0.006$, $f(3.79) = 0.001$. In view *of the errors of experiment the value 3.79 for k is close enough.* Using this value of k we have

$$\lambda_0 + \frac{2}{3}\mu_0 = \frac{kp_2}{\left(\frac{V_0}{V_2}\right)^k - 1} = \frac{22.74}{3.242} = 7.014$$

The formula (of the integrated linear theory) that furnishes $\frac{V}{V_0}$ is, accordingly,

$$\left(\frac{V_0}{V}\right)^{3.79} = \frac{3.79}{7.014}\left(p + \frac{7.014}{3.79}\right) = (0.540)(p + 1.851)$$

The following table gives the comparison between the values of $\frac{V}{V_0}$ computed from this formula and the *experimentally measured* value of $\frac{V}{V_0}$

$p =$	2.5	3	4	5	6	7	8	9	10
$\frac{V}{V_0}$ (measured) =	0.789	0.770	0.737	0.708	0.683	0.661	0.641	0.623	0.606
$\frac{V}{V_0}$ (calculated) =	0.798	0.776	0.738	0.708	0.683	0.662	0.643	0.627	0.613

The greatest divergence between the calculated and the measured values is at $p = 2.5$, where the calculated value is in excess of the

measured value by less than 1.5 per cent. Since the experimental values do not claim an accuracy of more than 2 or 3 per cent we may say that the values furnished by the two-constant formula of the integrated linear theory check the experimentally measured values over the entire range of pressures from $p = 25000$ atmospheres to $p = 100{,}000$ atmospheres.

This check between theory and experiment gives us confidence to predict the values of $\frac{V}{V_0}$ that will be measured when the experimental technique is sufficiently developed that measurements of $\frac{V}{V_0}$ up to $p = 20$ (i.e., up to a pressure of 200,000 atmospheres) can be made. We use the experimentally measured values of $\frac{V}{V_0}$ at $p = 9$ and at $p = 10$ to redetermine the values of k and $\lambda_0 + \frac{2}{3}\mu_0$. The equation that serves to determine k is now

$$10\left(\frac{V_0}{V_1}\right)^k - 9\left(\frac{V_0}{V_2}\right)^k - 1 = 0$$

where $\frac{V_1}{V_0} = .623$, $\frac{V_2}{V_0} = .606$. We find that (perform the calculation or verify the result) $k = 2.86$, $\lambda_0 + \frac{2}{3}\mu_0 = 8.968$. *Note.* The difference between these values for k and $\lambda_0 + \frac{2}{3}\mu_0$ and those obtained previously (when p_1 was 5 and p_2 was 6) merely indicates the difference between the linear approximation to $\lambda + \frac{2}{3}\mu$ in the interval $9 \leqq p \leqq 10$ and the linear approximation to $\lambda + \frac{2}{3}\mu$ in the interval $5 \leqq p \leqq 6$. The formula determining $\frac{V}{V_0}$ is, now,

$$\left(\frac{V_0}{V}\right)^{2.86} = (0.319)(p + 3.135)$$

and the comparison between the experimentally determined values and the computed values over the range $p = 2.5, 3, \cdots, 10$, together with the predicted values over the range $p = 11, \cdots, 20$, is furnished by the table on p. 76.

Thus the theory predicts that a pressure of 200,000 atmospheres will compress sodium to 0.497 of its original volume. At a pressure of 1,000,000 atmospheres the formula yields $\frac{V}{V_0} = 0.400$, and if we have confidence enough in the formula to use it at $p = 100$ we find that the corresponding value of $\frac{V}{V_0}$ is 0.295. In other words, a pressure

of 10^6 atmospheres will compress sodium to less than three-tenths its original volume. The agreement between the values of $\frac{V}{V_0}$ computed from the formula and the experimentally measured values, is within 3 per cent over the entire range $p = 2.5, 3, \ldots, 10$ of the experiments. The greatest discrepancy is at the pressure 2.5, where the calculated value of $\frac{V}{V_0}$ is 0.815 and the measured value 0.789. This measured value is somewhat uncertain as it was obtained by an adjustment between two series of experiments with different apparatus. Over the range $p = 4$ to $p = 10$, for which direct readings from one apparatus were obtained, the agreement between the values given by the formulas and the experimentally measured values is within 2 per cent (the discrepancies for all the values save the first being less than 1 per cent).

$p =$	2.5	3	4	5	6	7	8	9	10
$\frac{V}{V_0}$ (measured)	0.789	0.770	0.737	0.708	0.683	0.661	0.641	0.623	0.606
$\frac{V}{V_0}$ (calculated)	0.815	0.782	0.750	0.715	0.688	0.664	0.642	0.623	0.606

$p =$	11	12	13	14	15	16	17	18	19	20
$\frac{V}{V_0}$ (calculated)	0.591	0.577	0.564	0.552	0.541	0.531	0.522	0.513	0.505	0.497

Remark. A noteworthy feature of the formula connecting p and $\frac{V}{V_0}$ which is furnished by the integrated linear theory is the unsymmetrical way in which it treats *pressure* ($p > 0$) and hydrostatic tension ($p < 0$). Thus it is clear from the formula

$$p = \frac{(\lambda_0 + \frac{2}{3}\mu_0)}{k}\left\{\left(\frac{V_0}{V}\right)^k - 1\right\}$$

that, as $V \to 0$, $p \to \infty$ and, as $V \to \infty$, $p \to -\frac{(\lambda_0 + \frac{2}{3}\mu_0)}{k}$. In other words, *the medium cannot, according to the integrated linear theory, support, without rupture, a hydrostatic tension of amount* $\frac{\lambda_0 + \frac{2}{3}\mu_0}{k}$, although it can support an arbitrarily large hydrostatic

pressure. From the formula in which the constants (for sodium) were determined by using the experimentally measured values of $\dfrac{V}{V_0}$ at $p_1 = 5$ and $p_2 = 6$ it would appear that the medium cannot support a hydrostatic tension of 1.851×10^4 atmospheres, and when the constants are calculated from the experimentally measured values of $\dfrac{V}{V_0}$ at $p_1 = 9$ and $p_2 = 10$ this *rupturing* hydrostatic tension is raised to 3.135×10^4 atmospheres.

We now turn to the formula furnished by the second-order approximation:

$$p = \left(\frac{\rho_z}{\rho_0}\right)^{1/3} \{(3\lambda_0 + 2\mu_0)e - (9l_0 + n_0)e^2\}$$

$$= \left(\frac{V}{V_0}\right)^{-1/3} \{\alpha e + \beta e^2\}; \qquad \alpha = 3\lambda_0 + 2\mu_0,\ \beta = -(9l_0 + n_0).$$

Here $1 - 2e = \left(\dfrac{V}{V_0}\right)^{2/3}$, and so the data on sodium may be presented as follows:

$p =$	2.5	3	4	5	6	7	8	9	10
$\dfrac{V}{V_0} =$	0.789	0.770	0.737	0.708	0.683	0.661	0.641	0.623	0.606
$e =$	0.0731	0.0799	0.0920	0.1028	0.1122	0.1206	0.1283	0.1353	0.1420
$p\left(\dfrac{V}{V_0}\right)^{1/3} =$	2.310	2.750	3.613	4.456	5.284	6.098	6.898	7.687	8.462

Denoting, for a moment, by q the product of p by $\left(\dfrac{V}{V_0}\right)^{1/3}$, we have the following two equations to determine the two constants α and β of our formula:

$$\alpha e_1 + \beta e_1^2 = q_1,$$

$$\alpha e_2 + \beta e_2^2 = q_2,$$

e_1 and e_2 being the values of e which correspond to any two values p_1 and p_2, respectively, of p. The formulas that serve to determine α and β are, accordingly,

$$\alpha = \frac{q_1 e_2^2 - q_2 e_1^2}{e_1 e_2(e_2 - e_1)}, \qquad \beta = \frac{q_2 e_1 - q_1 e_2}{e_1 e_2(e_2 - e_1)}.$$

In order to test the value of the second-order approximation for the purposes of prediction we again suppose that we are in possession of experimental measurements up to $p = 6$ only and we use the formula to *predict the values of* $\frac{V}{V_0}$ *corresponding to* $p = 7, 8, 9,$ *and* 10 Since α and β are very sensitive to small variations in e_1 and e_2 if $e_2 - e_1$ is small, we use the values $p_1 = 2\ 5$ and $p_2 = 6$ in determining α and β We find (check this calculation) that

$$\alpha = 2\ 619, \qquad \beta = 396\ 3$$

The results of the comparison between theory and experiment is furnished by the following table

p(observed) =	2 5	3	4	5	6	7	8	9	10
$q = p\left(\frac{V}{V_0}\right)^{1/3}$ (observed) =	2 310	2 750	3 613	4 456	5 284	6 098	6 898	7 687	8 462
q(calculated) =	2 309	2 739	3 595	4 457	5 282	6 080	6 859	7 608	8 362
p(calculated) =	2 499	2 981	3 980	5 001	5 998	6 980	7 956	8 908	9 882

Note For reasons of convenience we have calculated the values of p that would correspond (when we use the formula of the second order approximation) to the experimentally measured values of $\frac{V}{V_0}$ rather than, as with the formula of the integrated linear theory, the values of $\frac{V}{V_0}$ that would correspond to the experimentally controlled values of p The greatest divergence between the calculated value of p and the experimentally controlled value of p occurs when the experimentally controlled value is 10 and the calculated value is less than this by less than 1 2 per cent Since the experiments do not claim an accuracy of more than 2 per cent, we may say that the formula of the second order approximation yields correctly the connection between p and $\frac{V}{V_0}$ up to $p = 10$, i e, up to a pressure almost twice as great as the larger of the two pressures that were used in determining the constants of the formula

If we determine the constants α and β by means of the *experimental measurements at* $p_1 = 2\ 5$, $p_2 = 10$, *instead of by means of the* experimental measurements at $p_1 = 2\ 5$, $p_2 = 6$, we obtain the

following values:

$$\alpha = 1.902, \qquad \beta = 406.2.$$

The comparison between the experimentally measured values and the values obtained from the formula

$$p\left(\frac{V}{V_0}\right)^{\frac{1}{3}} = 1.902c + 406.2c^2$$

is furnished by the following table:

p (observed)	2.5	3	4	5	6	7	8	9	10
$q = p\left(\frac{V}{V_0}\right)^{\frac{1}{3}}$ (observed)	2.310	2.750	3.613	4.456	5.284	6.098	6.898	7.687	8.462
q (calculated)	2.310	2.746	3.615	4.489	5.327	6.138	6.931	7.694	8.462
p (calculated)	2.500	2.996	4.002	5.037	6.049	7.046	8.038	9.008	10.00

Thus the calculated values of p differ from the observed values of p by less than 1 per cent (i.e., by less than the experimental margin of error) over the entire range from $p = 1$ to $p = 10$.

To find the value of p which would correspond to a given value, say, 0.497 of $\frac{V}{V_0}$ we proceed as follows: From $\left(\frac{V}{V_0}\right)^{\frac{2}{3}} = 0.6275$ we obtain $c = 0.1862$ and so $p\left(\frac{V}{V_0}\right)^{\frac{1}{3}} = 14.44$, $p = 18.23$ (as compared with the value $p = 20$ furnished by the extended linear theory).

Note. The values 2.619 and 396.3 for α and β, respectively, determined from the experimental observations at $p_1 = 2.5$ and $p_2 = 6$ differ from the values 1.902 and 406.2, respectively, determined from the measurements at $p_1 = 2.5$ and $p_2 = 10$. This merely indicates that the assumption that q is a quadratic function of c over the range $0 \leqslant p \leqslant 10$ is only an approximation to the truth; the *parabolic approximation* over the range $2.5 \leqslant p \leqslant 6$ differs slightly from the *parabolic approximation* over the range $2.5 \leqslant p \leqslant 10$. The fact that β is so large compared with α indicates the importance of considering the third-order elastic constants. Whereas $\lambda_0 + \frac{2}{3}\mu_0$ is around 0.7, $9l + n$ is negative and around -400.

In order to test the value of the second order approximation for the purposes of prediction we again suppose that we are in possession of experimental measurements up to $p = 6$ only and we use the formula to predict the values of $\frac{V}{V_0}$ corresponding to $p = 7, 8, 9,$ and 10 Since α and β are very sensitive to small variations in e_1 and e_2 if $e_2 - e_1$ is small, we use the values $p_1 = 2\ 5$ and $p_2 = 6$ in determining α and β We find (check this calculation) that

$$\alpha = 2\ 619, \qquad \beta = 396\ 3$$

The results of the comparison between theory and experiment is furnished by the following table

p(observed) =	2 5	3	4	5	6	7	8	9	10
$q = p\left(\frac{V}{V_0}\right)^{1/4}$ (observed) =	2 310	2 750	3 613	4 456	5 284	6 098	6 898	7 687	8 462
q(calculated) =	2 309	2 739	3 595	4 457	5 282	6 080	6 859	7 608	8 362
p(calculated) =	2 499	2 981	3 980	5 001	5 998	6 980	7 956	8 903	9 882

Note For reasons of convenience we have calculated the values of p that would correspond (when we use the formula of the second order approximation) to the experimentally measured values of $\frac{V}{V_0}$ rather than, as with the formula of the integrated linear theory, the values of $\frac{V}{V_0}$ that would correspond to the experimentally controlled values of p The greatest divergence between the calculated value of p and the experimentally controlled value of p occurs when the experimentally controlled value is 10 and the calculated value is less than this by less than 1 2 per cent Since the experiments do not claim an accuracy of more than 2 per cent we may say that the formula of the second order approximation yields correctly the connection between p and $\frac{V}{V_0}$ up to $p = 10$, i e , up to a pressure almost twice as great as the larger of the two pressures that were used in determining the constants of the formula

If we determine the constants α and β by means of the experi*mental measurements at* $p_1 = 2\ 5$ $p_2 = 10$, *instead of by means of* the experimental measurements at $p_1 = 2\ 5$, $p_2 = 6$, we obtain the

following values:

$$\alpha = 1.902, \qquad \beta = 406.2.$$

The comparison between the experimentally measured values and the values obtained from the formula

$$p\left(\frac{V}{V_0}\right)^{1/3} = 1.902e + 406.2e^2$$

is furnished by the following table:

p (observed)	2.5	3	4	5	6	7	8	9	10
$q = p\left(\frac{V}{V_0}\right)^{1/3}$ (observed)	2.310	2.750	3.613	4.456	5.281	6.098	6.898	7.687	8.462
q (calculated)	2.310	2.746	3.615	4.489	5.327	6.138	6.931	7.694	8.462
p (calculated)	2.500	2.996	4.002	5.037	6.019	7.016	8.038	9.008	10.00

Thus the calculated values of p differ from the observed values of p by less than 1 per cent (i.e., by less than the experimental margin of error) over the entire range from $p = 1$ to $p = 10$.

To find the value of p which would correspond to a given value, say, 0.497 of $\frac{V}{V_0}$ we proceed as follows: From $\left(\frac{V}{V_0}\right)^{2/3} = 0.6275$ we obtain $e = 0.1862$ and so $p\left(\frac{V}{V_0}\right)^{1/3} = 14.44$, $p = 18.23$ (as compared with the value $p = 20$ furnished by the extended linear theory).

Note. The values 2.619 and 396.3 for α and β, respectively, determined from the experimental observations at $p_1 = 2.5$ and $p_2 = 6$ differ from the values 1.902 and 406.2, respectively, determined from the measurements at $p_1 = 2.5$ and $p_2 = 10$. This merely indicates that the assumption that q is a quadratic function of e over the range $0 \leqslant p \leqslant 10$ is only an approximation to the truth; the *parabolic approximation* over the range $2.5 \leqslant p \leqslant 6$ differs slightly from the *parabolic approximation* over the range $2.5 \leqslant p \leqslant 10$. The fact that β is so large compared with α indicates the importance of considering the third-order elastic constants. Whereas $\lambda_0 + \frac{2}{3}\mu_0$ is around 0.7, 9l [illegible] is negative and around -400.

5

NON-ISOTROPIC ELASTIC MEDIA

1. Definition of a non-isotropic elastic medium

Certain materials (for example, wood) that are important in construction are not isotropic i e , they are sensitive to at least some rotations of the medium (before application of the strain) We have seen also that even if a medium is isotropic when it is initially free from stress or when the initial stress is a hydrostatic pressure or tension, it *cannot* be isotropic when the initial stress is any stress other than such a hydrostatic pressure or tension It is, therefore, highly important, from the scientific point of view, to obtain the relation connecting stress and strain for *non-isotropic* media Unfortunately, however, this relation is much more complicated than for isotropic media, even in the infinitesimal theory many more elastic constants than the two, λ and μ that suffice for isotropic media are needed In the second order approximation the number of elastic constants becomes, usually, so large that the theory is too complicated for practical applications, but it is possible to construct an *integrated linear theory* that is useful in practice Many deformable media have a *crystalline structure* so that, although they are not isotropic, i e , elastically insensitive to all prerotations of the medium they are elastically insensitive to *certain* such prerotations Thus the relation

$$\phi(R^*\eta R) = \phi(\eta)$$

will be now valid for certain given rotation matrices R but not for all rotation matrices When ϕ is analyzed into a sum

$$\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 +$$

of terms of different degrees in the elements of η (ϕ_j being a homogeneous function of degree j in the elements of η, $j = 0, 1, 2,$), the relation $\phi(R^*\eta R) = \phi(\eta)$ implies the series of relations

$$\phi_j(R^*\eta R) = \phi_j(\eta), \qquad j = 0, 1, 2$$

(Prove this *Hint* The relation $\phi(R^*\eta R) = \phi(\eta)$ is an identity in the elements of η It remains true, then, if η is replaced by $\lambda\eta$ where λ

is an arbitrary parameter; when this substitution is made $\phi_j(\eta)$ is replaced by $k^j\phi_j(\eta)$.) The relation $\phi_j(R^*\eta R) = \phi_j(\eta)$ imposes certain conditions on the coefficients of the homogeneous function, $\phi_j(\eta)$, of degree j, of the elements of η (save in the case $j = 0$ in which we are not interested [why?]) and we proceed to find the nature of these conditions, for various given rotation matrices R, when $j = 1$ or $j = 2$ or $j = 3$.

EXERCISES

1. Show that if a deformable medium is elastically insensitive to a given rotation R it is elastically insensitive to every integral power, positive or negative, of R. *Hint.* Replace η by $R^*\eta R$ and by $R\eta R^*$ in the relation $\phi(R^*\eta R) = \phi(\eta)$.

2. Show that if the medium is elastically insensitive to a given rotation R it is elastically insensitive to the *reflexion* $-R$. *Note.* A reflexion is an orthogonal transformation of rectangular Cartesian coordinates the determinant of whose matrix is -1 (instead of $+1$ as for a rotation).

3. Show that if R is a rotation through π around the a-axis, $-R$ is a reflexion in the a-plane.

4. Show that if the medium is elastically insensitive to two given rotations R_1 and R_2 it is elastically insensitive to their product R_1R_2.

5. Show that if the medium is elastically insensitive to a rotation through π around the a-axis and to a rotation through π around the b-axis it is elastically insensitive to a rotation through π around the c-axis. *Hint.* Any one of these rotations is the product of the other two.

6. Show that if the medium is elastically insensitive to a rotation through $\frac{\pi}{2}$ around the a-axis and to a rotation through $\frac{\pi}{2}$ around the b-axis it is elastically insensitive to a rotation through $\frac{\pi}{2}$ around the c-axis. *Hint.* Denoting these rotations by R_1, R_2, and R_3, respectively, $R_3 = R_1R_2R_1^*$. *Note.* Certain cubic crystals possess the elastic symmetry of this exercise.

7. Show that if a medium possesses the elastic symmetry described in Exercise 6 it is elastically insensitive to a rotation through $\frac{2\pi}{3}$ around each of the four diagonals of the cube whose vertices are the points $(\pm 1, \pm 1, \pm 1)$. *Hint.* The product of a rotation through $\frac{\pi}{2}$ around the b-axis by a rotation through $\frac{\pi}{2}$ around the c-axis is a rotation through $\frac{2\pi}{3}$ around one of the diagonals of the cube in question.

2. The form of $\phi_1(\eta)$ for various rotations R

$\phi_1(\eta)$ is a linear function of the elements of η. When written symmetrically (what does this mean?) it appears in the form

$$\phi_1(\eta) = Tr(C_1\eta)$$

where C_1 is a symmetric 3×3 matrix. The elements of C_1 are the values, when $\eta = 0$, of R^*TR for

$$R^*TR = \left(\frac{\rho_x}{\rho_a}\right) M^{1/2} \frac{\partial \phi}{\partial \eta} M^{1/2}$$

and when $\eta = 0$, $\dfrac{\rho_x}{\rho_a} = 1$, $M = E_3$, and $\dfrac{\partial \phi}{\partial \eta} = C_1$. Thus,

The coefficients of the linear function $\phi_1(\eta)$ of the elements of η, when this linear function is written symmetrically, are the coordinates of the initial stress in the reference frame in which $JJ^* = J^*J$. On denoting the initial stress matrix by T_0 we have

$$R^*T_0R = C_1, \qquad R = JM^{-1/2}$$

We now denote the variables $(\eta_{aa}, \eta_{bb}, \eta_{cc}, \eta_{bc}, \eta_{ca}, \eta_{ab})$ by $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)$ and write $\phi_1(\eta)$ in the form

$$\phi_1(\eta) = c_1\eta_1 + c_2\eta_2 + c_3\eta_3 + c_4\eta_4 + c_5\eta_5 + c_6\eta_6$$

Thus the symmetric 3×3 matrix C_1 is

$$C_1 = \begin{pmatrix} c_1 & \frac{1}{2}c_6 & \frac{1}{2}c_5 \\ \frac{1}{2}c_6 & c_2 & \frac{1}{2}c_4 \\ \frac{1}{2}c_5 & \frac{1}{2}c_4 & c_3 \end{pmatrix}$$

Under the rotation $a \to a' = R^*a$, $\eta \to \eta' = R^*\eta R$, and since $aa^* \to a'a'^* = R^*aa^*R$ it follows that the elements of η transform, under the rotation $a \to a' = R^*a$, like the elements of aa^*. This fact implies (show this) that $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6)$ transform like $(a^2, b^2, c^2, bc, ca, ab)$. Let us consider the case in which R is a rotation through an angle θ around the a-axis, since the a axis is arbitrary, this takes care of every rotation. There is no lack of generality in taking $0 < \theta \leq \pi$, since R may be replaced by R^* and we shall do so. We have

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

so that $a' = a$, $b' = (\cos\theta)b + (\sin\theta)c$, $c' = -(\sin\theta)b + (\cos\theta)c$. Hence $\eta \to \eta'$ where

$$\eta_1' = \eta_1, \qquad \eta_2' = (\cos^2\theta)\eta_2 + 2(\cos\theta\sin\theta)\eta_4 + (\sin^2\theta)\eta_3,$$

$$\eta_3' = (\sin^2\theta)\eta_2 - 2(\cos\theta\sin\theta)\eta_4 + (\cos^2\theta)\eta_3,$$

$$\eta_4' = -(\cos\theta\sin\theta)\eta_2 + (\cos^2\theta - \sin^2\theta)\eta_4 + (\cos\theta\sin\theta)\eta_3,$$

$$\eta_5' = (\cos\theta)\eta_5 - (\sin\theta)\eta_6, \qquad \eta_6' = (\sin\theta)\eta_5 + (\cos\theta)\eta_6$$

Since $\eta_2' + \eta_3' = \eta_2 + \eta_3$ it is convenient to introduce, as replacing the variables $(\eta_1, \cdots, \eta_6)$, the variables $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ where

$$\xi_1 = \eta_1, \quad \xi_2 = \tfrac{1}{2}(\eta_2 + \eta_3), \quad \xi_3 = \tfrac{1}{2}(\eta_2 - \eta_3),$$
$$\xi_4 = \eta_4, \quad \xi_5 = \eta_5, \quad \xi_6 = \eta_6.$$

Then $\eta_1 = \xi_1$, $\eta_2 = \xi_2 + \xi_3$, $\eta_3 = \xi_2 - \xi_3$, $\eta_4 = \xi_4$, $\eta_5 = \xi_5$, $\eta_6 = \xi_6$

and, under the rotation $a \to a' = R^*a$, $\xi \to \xi'$ where

$$\xi_1' = \xi_1, \quad \xi_2' = \xi_2, \quad \xi_3' = (\cos 2\theta)\xi_3 + (\sin 2\theta)\xi_4,$$
$$\xi_4' = -(\sin 2\theta)\xi_3 + (\cos 2\theta)\xi_4, \quad \xi_5' = (\cos \theta)\xi_5 - (\sin \theta)\xi_6,$$
$$\xi_6' = (\sin \theta)\xi_5 + (\cos \theta)\xi_6.$$

These relations suggest the introduction of the (complex) variables $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ defined by the equations

$$\zeta_1 = \xi_1, \quad \zeta_2 = \xi_2, \quad \zeta_3 = \tfrac{1}{2}(\xi_3 - i\xi_4),$$
$$\zeta_4 = \tfrac{1}{2}(\xi_3 + i\xi_4), \quad \zeta_5 = \tfrac{1}{2}(\xi_5 - i\xi_6), \quad \zeta_6 = \tfrac{1}{2}(\xi_5 + i\xi_6).$$

Of the six variables $(\zeta_1, \cdots, \zeta_6)$, ζ_1 and ζ_2 are real, and ζ_3 and ζ_4 are conjugate complex numbers as are also ζ_5 and ζ_6. It is clear that $\xi_1 = \zeta_1$, $\xi_2 = \zeta_2$, $\xi_3 = \zeta_3 + \zeta_4$, $\xi_4 = i(\zeta_3 - \zeta_4)$, $\xi_5 = \zeta_5 + \zeta_6$, $\xi_6 = i(\zeta_5 - \zeta_6)$ and it follows (show this) that, under the rotation $a \to a' = R^*a$, $\zeta \to \zeta'$ where

$$\zeta_1' = \zeta_1, \quad \zeta_2' = \zeta_2, \quad \zeta_3' = e^{2i\theta}\zeta_3, \quad \zeta_4' = e^{-2i\theta}\zeta_4,$$
$$\zeta_5' = e^{-i\theta}\zeta_5, \quad \zeta_6' = e^{i\theta}\zeta_6.$$

The relations connecting the original (real) variables $(\eta_1 \cdots \eta_6)$ with the new (complex) variables $(\zeta_1 \cdots \zeta_6)$ are

$$\eta_1 = \xi_1 = \zeta_1; \qquad \zeta_1 = \xi_1 = \eta_1;$$

$$\eta_2 = \xi_2 + \xi_3 = \zeta_2 + \zeta_3 + \zeta_4; \qquad \zeta_2 = \xi_2 = \tfrac{1}{2}(\eta_2 + \eta_3);$$

$$\eta_3 = \xi_2 - \xi_3 = \zeta_2 - \zeta_3 - \zeta_4; \qquad \zeta_3 = \frac{1}{2}(\xi_3 - i\xi_4) = \frac{1}{4}(\eta_2 - \eta_3) - \frac{i}{2}\eta_4;$$

$$\eta_4 = \xi_4 = i(\zeta_3 - \zeta_4); \qquad \zeta_4 = \frac{1}{2}(\xi_3 + i\xi_4) = \frac{1}{4}(\eta_2 - \eta_3) + \frac{i}{2}\eta_4;$$

$$\eta_5 = \xi_5 = \zeta_5 + \zeta_6; \qquad \zeta_5 = \tfrac{1}{2}(\xi_5 - i\xi_6) = \tfrac{1}{2}(\eta_5 - i\eta_6);$$

$$\eta_6 = \xi_6 = i(\zeta_5 - \zeta_6); \qquad \zeta_6 = \tfrac{1}{2}(\xi_5 + i\xi_6) = \tfrac{1}{2}(\eta_5 + i\eta_6).$$

We denote the coefficients of $\phi_1(\eta)$, when this expression is written as a function of the variables $(\zeta_1, \cdots, \zeta_6)$, by $(d_1, \cdots, d_6)$ and so

$\phi_1(\eta) = c_1\eta_1 + c_2\eta_2 + c_3\eta_3 + c_4\eta_4 + c_5\eta_5 + c_6\eta_6 = d_1\zeta_1 + d_2\zeta_2 + d_3\zeta_3 + d_4\zeta_4 + d_5\zeta_5 + d_6\zeta_6$ d_1 and d_2 are real, and d_3 and d_4 are conjugate complex numbers as are also d_5 and d_6 The necessary and sufficient condition that the medium should be elastically insensitive to the rotation R, as far as ϕ_1 is concerned, is that the relation

$$d_1\zeta_1' + \qquad + d_6\zeta_6' = d_1\zeta_1 + \qquad + d_6\zeta_6$$

should be an identity in the six variables $(\zeta_1, \quad, \zeta_6)$ Upon substituting for $(\zeta_1', \quad, \zeta_6')$ their expressions in terms of $(\zeta_1, \quad, \zeta_6)$, we obtain the following relations involving $(d_1, \quad, d_6)$ and the angle θ of the rotation R

$$e^{2i\theta}d_3 = d_3, \qquad e^{-i\theta}d_5 = d_5$$

Note In writing these equations we have put down only one of each pair of conjugate complex equations, thus the equation $e^{2i\theta}d_3 = d_3$ implies the equation $e^{-2i\theta}d_4 = d_4$, and the equation $e^{-i\theta}d_5 = d_5$ implies the equation $e^{i\theta}d_6 = d_6$

If θ is not the quotient of 2π by 2, i e, if $\theta \neq \pi$, neither of the numbers $e^{2i\theta}$, $e^{-i\theta}$ is unity and we have $d_3 = 0$ $d_5 = 0$ The medium is then elastically insensitive, as far as ϕ_1 is concerned, to a rotation through *any* angle around the a axis (why?) and $\phi_1(\eta)$ is a linear combination of $\zeta_1 = \eta_1$ and $\zeta_2 = \frac{1}{2}(\eta_2 + \eta_3)$ We have, then, the following result

If a deformable medium is elastically insensitive to a rotation through an angle $\theta \neq \pi$ around the a-axis it is, as far as ϕ_1 is concerned, elastically insensitive to a rotation through any angle around the a-axis and $\phi_1(\eta)$ is a linear combination of η_1 and $\eta_2 + \eta_3$

$$\phi_1(\eta) = c_1\eta_1 + c_2(\eta_2 + \eta_3)$$

Thus the medium cannot be elastically insensitive to such a rotation unless the initial stress matrix is, in the reference frame in which $JJ^* = J^*J$, diagonal

EXERCISES

1 What is the necessary and sufficient condition that the medium be elastically insensitive as far as ϕ_1 is concerned, to a rotation through any angle around the b-axis? around the c-axis? *Hint* When the $(a\ b\ c)$-axes are subjected to a cyclic permutation the two sets (123) and (456) are subjected to the same cyclic permutation

2 Show that if the medium is elastically insensitive as far as ϕ_1 is concerned to a rotation through any angle around the a axis and to a rotation through any angle around the b axis it is as far as ϕ_1 is concerned, isotropic *Hint* $\phi_1(\eta)$ is of the form $c(\eta_1 + \eta_2 + \eta_3) = cI_1$ *Note* The phrase 'as far as ϕ_1 is concerned' may be omitted since every rotation may be factored into the product of three rotations, two of which are about the a axis the third being about the b axis

If $\theta = \pi$, $e^{2i\theta} = 1$ and d_3 is arbitrary (instead of being zero). $\phi_1(\eta)$ is now a linear combination of ζ_1, ζ_2, ζ_3, and ζ_4 or, equivalently, of η_1, η_2, η_3, η_4:

$$\phi_1(\eta) = c_1\eta_1 + c_2\eta_2 + c_3\eta_3 + c_4\eta_4.$$

The initial stress matrix (in the reference frame in which $JJ^* = J^*J$) is of the form

$$\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & \frac{1}{2}c_4 \\ 0 & \frac{1}{2}c_4 & c_3 \end{pmatrix}.$$

The necessary and sufficient condition that the medium should be elastically insensitive, as far as ϕ_1 is concerned, to a rotation through π around the a-axis or, equivalently (why?), to a reflexion in the a-plane, is that $\phi_1(\eta) = c_1\eta_1 + c_2\eta_2 + c_3\eta_3 + c_4\eta_4$.

EXERCISES

3. What is the necessary and sufficient condition that the medium be elastically insensitive, as far as ϕ_1 is concerned, to a reflexion in the b-plane? in the c-plane?

4. Show that if the medium is elastically insensitive, as far as ϕ_1 is concerned, to a rotation through an angle $\theta \neq \pi$ around the a-axis it is elastically insensitive, as far as ϕ_1 is concerned, to a reflexion in each of the coordinate planes.

3. The form of $\phi_2(\eta)$ for various rotations R

$2\phi_2(\eta)$ is a homogeneous quadratic function of the elements of η, and so it may be written in the form

$$2\phi_2(\eta) = (c_{11}\eta_1^2 + \cdots + c_{66}\eta_6^2) + 2(c_{12}\eta_1\eta_2 + \cdots + c_{56}\eta_5\eta_6).$$

The medium is elastically insensitive, as far as ϕ_2 is concerned, to a given rotation R if, and only if, the relation

independent, in this event we say that the medium possesses fewer than twenty-one second order elastic constants An isotropic medium possesses only two second-order elastic constants, the degree of complexity introduced into the theory of elasticity by lack of isotropy may be grasped when we compare this with the twenty-one second-order elastic constants of a medium that is completely non isotropic (as far as ϕ_2 is concerned)

We now consider the case in which R is a rotation through an angle θ around the a-axis, and we introduce the complex variables $(\zeta_1, \cdots, \zeta_6)$ of the preceding section We denote the coefficients of $2\phi_2(\eta)$, when written as a function of $(\zeta_1, \cdots, \zeta_6)$, by d_{pq}

$$2\phi_2(\eta) = d_{11}\zeta_1^2 + \cdots + d_{66}\zeta_6^2 + 2(d_{12}\zeta_1\zeta_2 + \cdots + d_{56}\zeta_5\zeta_6)$$

Here d_{11} and d_{22} are real, and d_{33} and d_{44} are conjugate complex numbers as are also d_{55} and d_{66}, d_{12}, d_{34}, and d_{56} are real, d_{13} and d_{14} are conjugate complex numbers, etc The necessary and sufficient condition that the medium be elastically insensitive, as far as $\phi_2(\eta)$ is concerned, to the rotation R is that the relation

$$\begin{aligned} d_{11}(\zeta_1')^2 + \cdots + d_{66}(\zeta_6')^2 + 2(d_{12}\zeta_1'\zeta_2' + \cdots + d_{56}\zeta_5'\zeta_6') \\ = d_{11}\zeta_1^2 + \cdots + d_{66}\zeta_6^2 + 2(d_{12}\zeta_1\zeta_2 + \cdots + d_{56}\zeta_5\zeta_6) \end{aligned}$$

should be an *identity in the six (complex) variables* $(\zeta_1, \cdots, \zeta_6)$ This furnishes the following relations involving the coefficients d_{pq} and the angle θ of the rotation R

$$\begin{aligned} &e^{4i\theta}d_{33} = d_{33}, \qquad e^{2i\theta}d_{13} = d_{13}, \qquad e^{-i\theta}d_{25} = d_{25}, \\ &e^{-2i\theta}d_{55} = d_{55}, \qquad e^{-i\theta}d_{15} = d_{15}, \qquad e^{i\theta}d_{35} = d_{35}, \\ &\qquad\qquad\qquad\quad e^{2i\theta}d_{23} = d_{23}, \qquad e^{3i\theta}d_{36} = d_{36} \end{aligned}$$

Note In writing these equations we have put down only one of each pair of conjugate complex equations, thus the equation $e^{4i\theta}d_{33} = d_{33}$ implies the equation $e^{-4i\theta}d_{44} = d_{44}$ etc

If θ is not the quotient of 2π by 2, 3, or 4 none of the multipliers $e^{4i\theta}$, $e^{\pm 2i\theta}$, $e^{\pm i\theta}$, $e^{3i\theta}$ is unity and so all the (complex) numbers d_{33}, d_{55}, d_{13}, d_{15}, d_{23}, d_{25}, d_{35}, d_{36} are zero In this event the medium is elastically insensitive to *every* rotation around the a axis and it possesses at most $21 - 2(8) = 5$ second-order elastic constants, $\phi_2(\eta)$ being a linear combination of ζ_1^2, ζ_2^2, $\zeta_1\zeta_2$, $\zeta_3\zeta_4$, and $\zeta_5\zeta_6$ or, equivalently, of η_1^2, $(\eta_2 + \eta_3)^2$, $\eta_1(\eta_2 + \eta_3)$, $(\eta_2 - \eta_3)^2 + 4\eta_4^2$, and $\eta_5^2 + \eta_6^2$ Since $\eta_1(\eta_2 + \eta_3) = I_1\eta_1 - \eta_1^2$, $I_1\eta_1$ may be substituted for $\eta_1(\eta_2 + \eta_3)$ in this linear combination, and since $\eta_2 + \eta_3 = I_1 - \eta_1$, I_1^2 may be substituted for $(\eta_2 + \eta_3)^2$ Since $(\eta_2 - \eta_3)^2 + 4\eta_4^2 = (\eta_2 + \eta_3)^2 -$

$4(\eta_2\eta_3 - \eta_4^2)$, $\eta_2\eta_3 - \eta_4^2$ may be substituted for $(\eta_2 - \eta_3)^2 + 4\eta_4^2$, and, finally, since $I_2 = (\eta_2\eta_3 - \eta_4^2) + \eta_1(\eta_2 + \eta_3) - (\eta_5^2 + \eta_6^2)$, I_2 may be substituted for $\eta_5^2 + \eta_6^2$. We have, then, the following result:

If a deformable medium is elastically insensitive (as far as ϕ_2 is concerned) to a rotation through an angle other than π, $\frac{2\pi}{3}$, or $\frac{\pi}{2}$ around the a-axis, it is elastically insensitive (as far as ϕ_2 is concerned) to every rotation around the a-axis. Such a medium possesses not more than five (instead of twenty-one) second-order elastic constants, and $\phi_2(\eta)$ is a linear combination of the following five functions:

$$I_1^2,\ I_2,\ \eta_1^2,\ I_1\eta_1,\ \eta_2\eta_3 - \eta_4^2.$$

$\eta_1, \eta_2, \eta_3, \eta_4$ (involving ten coefficients) and a linear combination of η_5^2, η_6^2, and $\eta_5\eta_6$

7. What is the form of $\phi_2(\eta)$ if the medium is elastically insensitive to a rotation through π around the b axis (or, equivalently, to a reflexion in the b plane)? to a rotation through π around the c axis (or, equivalently, to a reflexion in the c plane)?

8 Show that if the medium is elastically insensitive to a rotation through π around the a axis and to a rotation through π around the b axis it possesses not more than nine second-order elastic constants, $\phi_2(\eta)$ being the sum of a general polynomial of the second degree in the three variables η_1, η_2, η_3 (involving six coefficients) and a linear combination of η_4^2, η_5^2, and η_6^2

When $\theta = \frac{2\pi}{3}$, the multiplier $e^{3i\theta}$ is unity and so d_{36} is arbitrary, instead of being zero The medium possesses (at most) $5 + 2 = 7$ second order elastic constants, and $\phi_2(\eta)$ is a linear combination of the five functions I_1^2, I_2, η_1^2, $I_1\eta_1$, $\eta_2\eta_3 - \eta_4^2$ and, in addition, the two functions $(\eta_2 - \eta_3)\eta_5 + 2\eta_4\eta_6$, $(\eta_2 - \eta_3)\eta_6 - 2\eta_4\eta_5$ (which are, respectively, the real and imaginary parts of $8\zeta_3\zeta_6$)

EXERCISE

9. Show that if a medium which possesses the elastic symmetry described in Exercise 8 is elastically insensitive to a rotation through $\frac{2\pi}{3}$ around the line joining the points (0, 0, 0) and (1, 1, 1) it possesses only three second-order elastic constants and that for such a medium $\phi_2(\eta)$ is a linear combination of I_1^2, I_2 and $(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2)$ *Hint* $\phi_2(\eta)$ is invariant under the transformation $\eta \to \eta' = R^*\eta R$ where

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

thus $\phi_2(\eta)$ is invariant under the transformation $\eta_1' = \eta_2$ $\eta_2' = \eta_3$, $\eta_3' = \eta_1$, $\eta_4' = \eta_5$, $\eta_5' = \eta_6$ $\eta_6' = \eta_4$ *Note* Certain cubic crystals possess the elastic symmetry of this exercise

When $\theta = \frac{\pi}{2}$, $e^{4i\theta}$ is unity and so d_{33} is arbitrary, instead of being zero The medium possesses (at most) $5 + 2 = 7$ second order elastic constants, and $\phi_2(\eta)$ is a linear combination of the five functions I_1^2, I_2, η_1^2, $I_1\eta_1$, $\eta_2\eta_3 - \eta_4^2$ and, in addition, the two functions $(\eta_2 - \eta_3)^2 - 4\eta_4^2$, $(\eta_2 - \eta_3)\eta_4$ (which are, respectively, convenient multiples of the real and imaginary parts of ζ_3^2) It is easy to see that $\eta_2\eta_3$ and η_4^2 are insensitive to a rotation through $\frac{\pi}{2}$ around the a axis (verify this), and they may be substituted for $\eta_2\eta_3 - \eta_4^2$ and $(\eta_2 - \eta_3)^2 - 4\eta_4^2$ Thus we have the following result

The medium is elastically insensitive (as far as ϕ_2 is concerned) to a rotation through $\frac{\pi}{2}$ around the a-axis if, and only if, $\phi_2(\eta)$ is a linear combination of the following seven functions:

$$I_1^2,\ I_2,\ \eta_1^2,\ I_1\eta_1,\ \eta_2\eta_3,\ \eta_4^2,\ (\eta_2 - \eta_3)\eta_4.$$

EXERCISES

$$\phi_3 = \tfrac{1}{3}\{(c_{111}\eta_1^3 + \quad\quad) + 3(c_{112}\eta_1^2\eta_2 + \quad\quad) + 6(c_{123}\eta_1\eta_2\eta_3 + \quad\quad)\}$$

There are fifty-six coefficients c_{pqr}, $p \leqslant q \leqslant r$, in all six of the type c_{111}, thirty of the type c_{112}, twenty of the type c_{123} Thus if $\phi_3(\eta)$ is sensitive to every three-dimensional rotation R the medium possesses (as far as ϕ_3 is concerned) fifty-six elastic constants, we term these *third-order* elastic constants A medium that is completely non isotropic as far as ϕ_2 and ϕ_3 are concerned possesses, then, $21 + 56 = 77$ elastic constants of which twenty-one are second-order constants (i e, coefficients of ϕ_2) and fifty-six are third-order constants (i e, coefficients of ϕ_3) The six coefficients of ϕ_1 (or first-order elastic constants) are the coordinates (in the reference frame in which $JJ^* = J^*J$) of the initial stress We proceed to investigate what reduction is caused in the number of the third-order elastic constants when the medium is elastically insensitive to one or more given rotations

Introducing the complex variables $(\zeta_1, \quad , \zeta_6)$ of the preceding sections, we denote the coefficients of $3\phi_3(\eta)$, when this is written as a function of the variables $(\zeta_1, \quad , \zeta_6)$, by d_{pqr}, $p \leqslant q \leqslant r = 1, \quad , 6$

$$3\phi_3(\eta) = d_{111}\zeta_1^6 + \quad\quad + 3(d_{112}\zeta_1^2\zeta_2 + \quad\quad) + 6(d_{123}\zeta_1\zeta_2\zeta_3 + \quad\quad)$$

Of the fifty-six coefficients d_{pqr} eight are real (why?)—d_{111}, d_{222} d_{112}, d_{122}, d_{134}, d_{156}, d_{234}, d_{256}—and the rest are pairs of conjugate complex numbers (why?)

$$
\begin{array}{llll}
(d_{333}, d_{444}), & (d_{555}, d_{666}), & (d_{113}, d_{114}), & (d_{115}, d_{116}), \\
(d_{223}, d_{224}), & (d_{225}, d_{226}), & (d_{133}, d_{144}), & (d_{233}, d_{244}), \\
(d_{334}, d_{344}), & (d_{335}\ d_{446}), & (d_{336}, d_{445}), & (d_{155}, d_{166}), \\
(d_{255}, d_{266}), & (d_{355}, d_{466}), & (d_{455}, d_{366}), & (d_{556}, d_{566}), \\
(d_{123}, d_{124}), & (d_{125}, d_{126}), & (d_{135}, d_{146}), & (d_{136}, d_{145}), \\
(d_{235}, d_{246}), & (d_{236}, d_{245}), & (d_{345}, d_{346}), & (d_{356}\ d_{456})
\end{array}
$$

The necessary and sufficient condition that the medium be elastically insensitive, as far as ϕ_3 is concerned, to the rotation R through an angle θ around the a-axis is that the following relation should be an identity in the six complex variables $(\zeta_1, \quad , \zeta_6)$

$$d_{111}\zeta_1'^3 + \quad\quad + 3(d_{112}\zeta_1'^2\zeta_2' + \quad\quad) + 6(d_{123}\zeta_1'\zeta_2'\zeta_3' + \quad\quad)$$
$$= d_{111}\zeta_1^3 + \quad\quad + 3(d_{112}\zeta_1^2\zeta_2 + \quad\quad) + 6(d_{123}\zeta_1\zeta_2\zeta_3 + \quad\quad)$$

EXERCISES

1 Verify that each of the ten functions written above is insensitive to a rotation, through an arbitrary angle θ about the a axis *Hint* $\eta_1 = \zeta_1$, $\eta_2 + \eta_3 = 2\zeta_2$ $(\eta_2 - \eta_3)^2 + 4\eta_4^2 = 16\zeta_3\zeta_4$ $\eta_5^2 + \eta_6^2 = 4\zeta_5\zeta_6$ $(\eta_2 - \eta_3)(\eta_5^2 - \eta_6^2) - 4\eta_4\eta_5\eta_6 =$ *real part of* $16\zeta_3\zeta_5^2$, *and* $(\eta_2 - \eta_3)\eta_5\eta_6 + \eta_4(\eta_5^2 - \eta_6^2) =$ *imaginary part of* $-8\zeta_3\zeta_5^2$

2 Show that $\eta_2\eta_3 - \eta_4^2$ is insensitive to a rotation through an arbitrary angle θ around the a axis *Hint* $4(\eta_2\eta_3 - \eta_4^2) = (\eta_2 + \eta_3)^2 - \{(\eta_2 - \eta_3)^2 + 4\eta_4^2\}$

3 Show that $\eta_2\eta_6^2 + \eta_3\eta_5^2 + 2\eta_4\eta_5\eta_6$ is insensitive to a rotation through an arbitrary angle θ around the a axis *Hint* $2(\eta_2\eta_6^2 + \eta_3\eta_5^2 + 2\eta_4\eta_5\eta_6) = (\eta_2 + \eta_3)(\eta_5^2 + \eta_6^2) - \{(\eta_2 - \eta_3)(\eta_5^2 - \eta_6^2) - 4\eta_4\eta_5\eta_6\}$

4 Show that $I_3 = \eta_1(\eta_2\eta_3 - \eta_4^2) + (\eta_2\eta_6^2 + \eta_3\eta_5^2 + 2\eta_4\eta_5\eta_6) - (\eta_2 + \eta_3)(\eta_5^2 + \eta_6^2)$

It follows from the result of Exercise 2 that $\eta_2\eta_3 - \eta_4^2$ may be used as a substitute for $(\eta_2 - \eta_3)^2 + 4\eta_4^2$ and from the results of Exercises 3 and 4 that I_3 may be used as a substitute for $(\eta_2 - \eta_3)(\eta_5^2 - \eta_6^2) - 4\eta_4\eta_5\eta_6$ $I_1(\eta_2\eta_3 - \eta_4^2)$ may be used as a substitute for $(\eta_2 + \eta_3)(\eta_2\eta_3 - \eta_4^2)$ (why?), and $I_1(\eta_5^2 + \eta_6^2)$ may be used as a substitute for $(\eta_2 + \eta_3)(\eta_5^2 + \eta_6^2)$ (why?) Since $I_1 I_2 = I_1(\eta_2\eta_3 - \eta_4^2) - I_1(\eta_5^2 + \eta_6^2) + I_1\eta_1(\eta_2 + \eta_3) = I_1(\eta_2\eta_3 - \eta_4^2) - I_1(\eta_5^2 + \eta_6^2) + \eta_1^2(\eta_2 + \eta_3) + \eta_1(\eta_2 + \eta_3)^2$, it follows that $I_1 I_2$ may be used as a substitute for $\eta_1(\eta_2 + \eta_3)^2$ and, since $I_1\eta_1^2 = \eta_1^3 + \eta_1^2(\eta_2 + \eta_3)$, that $I_1\eta_1^2$ may be used as a substitute for $\eta_1^2(\eta_2 + \eta_3)$ Finally, since $I_1^3 = \eta_1^3 + 3\eta_1^2(\eta_2 + \eta_3) + 3\eta_1(\eta_2 + \eta_3)^2 + (\eta_2 + \eta_3)^3$, I_1^3 may be used as a substitute for $(\eta_2 + \eta_3)^3$ Hence we have the following result

A deformable medium is elastically insensitive, as far as $\phi_3(\eta)$ *is concerned, to an arbitrary rotation around the a axis if, and only if,* $\phi_3(\eta)$ *is a linear combination of the following ten functions of* $(\eta_1, \ldots, \eta_6)$

$$I_1^3,\quad I_1 I_2,\quad I_3,\quad \eta_1^3,\quad I_1\eta_1^2,\quad \eta_1(\eta_2\eta_3 - \eta_4^2),\quad I_1(\eta_2\eta_3 - \eta_4^2),$$
$$\eta_1(\eta_5^2 + \eta_6^2),\quad I_1(\eta_5^2 + \eta_6^2),\quad \eta_5\eta_6(\eta_2 - \eta_3) + \eta_4(\eta_5^2 - \eta_6^2)$$

EXERCISE

5 What is the form of $\phi_3(\eta)$ if the medium is elastically insensitive to every rotation about the b-axis? about the c axis?

When the angle θ of the rotation around the a-axis to which the medium is elastically insensitive is the quotient of 2π by 2, 3, 4, 5, or 6, some of the multipliers $e^{6i\theta}, \ldots, e^{-2i\theta}$ are unity and so some of the twenty three coefficients $d_{333}, \ldots, d_{456}$ are arbitrary complex numbers (instead of having to be zero) The medium may possess, then, more than the ten third order elastic constants to which it would be limited if θ were different from $\pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{5}$, or $\frac{\pi}{3}$ The results for these various values are as follows (check these statements)

elastic constants, and $\phi_3(\eta)$ is a linear combination of the ten functions already written down and of 10 new functions that may be taken to be the real and imaginary parts of ζ_3^3, ζ_5^3, $\zeta_3^2\zeta_5$, $\zeta_1\zeta_3\zeta_6$, and $\zeta_2\zeta_3\zeta_6$ (or any convenient ten linearly independent combinations of these)

When $\theta = \dfrac{\pi}{2}$, the multipliers $e^{4i\theta}$ and $e^{-4i\theta}$ each are unity, and so the three complex numbers d_{133}, d_{233}, and d_{455} are arbitrary (instead of being zero) The medium possesses (at most) $10 + 2(3) = 16$ third-order elastic constants, and $\phi_3(\eta)$ is a linear combination of the ten functions of $(\eta_1, \quad, \eta_6)$ that we have already written down and of six new functions that may be taken to be real and imaginary parts of $\zeta_1\zeta_3^2$, $\zeta_2\zeta_3^2$, $\zeta_4\zeta_5^2$ However, the form of $\phi_3(\eta)$ is most easily derived as follows Since the medium is elastically insensitive to a rotation through π around the a axis (why?), $3\phi_3(\eta)$ is of the form

$$\begin{aligned}
3\phi_3(\eta) = {} & c_{111}\eta_1^3 + c_{222}\eta_2^3 + c_{333}\eta_3^3 + c_{444}\eta_4^3 \\
& + 3(c_{112}\eta_1^2\eta_2 + \qquad + c_{344}\eta_3\eta_4^2) + 6(c_{123}\eta_1\eta_2\eta_3 + \\
& + c_{234}\eta_2\eta_3\eta_4) + 3(c_{155}\eta_1 + c_{255}\eta_2 + c_{355}\eta_3 + c_{455}\eta_4)\eta_5^2 \\
& + 6(c_{156}\eta_1 + c_{256}\eta_2 + c_{356}\eta_3 + c_{456}\eta_4)\eta_5\eta_6 \\
& \qquad\qquad + 3(c_{166}\eta_1 + c_{266}\eta_2 + c_{366}\eta_3 + c_{466}\eta_4)\eta_6^2
\end{aligned}$$

Under the rotation through $\dfrac{\pi}{2}$ around the a axis, namely,

$$a' = a, \qquad b' = c, \qquad c' = -b,$$

$\eta \to \eta'$ where $\eta_1' = \eta_1$, $\eta_2' = \eta_3$, $\eta_3' = \eta_2$, $\eta_4' = -\eta_4$, $\eta_5' = -\eta_6$, $\eta_6' = \eta_5$, and since the relation $\phi_3(\eta') = \phi_3(\eta)$ must be an identity in the six variables $(\eta_1, \quad, \eta_6)$ we obtain the following relations

$$\begin{aligned}
& c_{114} = c_{156} = c_{234} = c_{444} = 0, \quad c_{112} = c_{113}, \quad c_{122} = c_{133}, \\
& \qquad c_{124} = c_{134}, \quad c_{155} = c_{166}, \quad c_{222} = c_{333}, \quad c_{223} = c_{233}, \\
& \qquad c_{224} = c_{334}, \quad c_{244} = c_{344}, \quad c_{255} = c_{366}, \quad c_{256} = -c_{356}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad c_{266} = c_{355}, \quad c_{455} = c_{466}
\end{aligned}$$

Thus $3\phi_3(\eta)$ is of the form

$$\begin{aligned}
3\phi_3(\eta) = {} & c_{111}\eta_1^3 + c_{222}(\eta_2^3 + \eta_3^3) + 3\{c_{112}\eta_1^2(\eta_2 + \eta_3) \\
& + c_{122}\eta_1(\eta_2^2 + \eta_3^2) + c_{223}\eta_2\eta_3(\eta_2 + \eta_3) + c_{224}\eta_4(\eta_2^2 - \eta_3^2) \\
& + c_{144}\eta_1\eta_4^2 + c_{244}\eta_4^2(\eta_2 + \eta_3)\} + 6\{c_{123}\eta_1\eta_2\eta_3 \\
& + c_{124}\eta_1\eta_4(\eta_2 - \eta_3) + c_{256}\eta_5\eta_6(\eta_2 - \eta_3) + c_{456}\eta_4\eta_5\eta_6\} \\
& + 3\{c_{155}\eta_1(\eta_5^2 + \eta_6^2) + c_{255}(\eta_2\eta_5^2 + \eta_3\eta_6^2) \\
& \qquad\qquad + c_{266}(\eta_2\eta_6^2 + \eta_3\eta_5^2) + c_{455}\eta_4(\eta_5^2 - \eta_6^2)\}
\end{aligned}$$

6

SIMPLE SHEAR AND TENSION

1. The modified stress matrix

We have seen that when a deformable medium is in a state of equilibrium in a strained position the stress matrix T is connected with the applied force per unit mass matrix F by means of the relation

$$(\mathrm{div}_x\, T)^* + \rho_x F = 0$$

and with the applied force-per-unit-area matrix f by means of the relation

$$T\, dS^x = dS_x f$$

(dS^x being the matrix element of area, and dS_x the scalar element of area, of the surface of the medium when deformed) Since the surface of the medium when deformed is not given (what is given being the surface of the medium when undeformed), we introduce the 3×3 matrix T_a, which is such that

$$T_a\, dS^a = T\, dS^x$$

We term this 3×3 matrix T_a the *modified stress matrix* Since $T = \left(\frac{\rho_x}{\rho_a}\right) J\, \frac{\partial \phi}{\partial \eta} J^*$ and since $dS^x = \left(\frac{\rho_a}{\rho_x}\right) (J^*)^{-1}\, dS^a$ (see Exercise 1, p 17), we have

$$T_a = J \frac{\partial \phi}{\partial \eta}$$

Note Although the stress matrix T is symmetric the modified stress matrix T_a is not, in general, symmetric T_a will be symmetric if J is symmetric and if the medium is isotropic, for, then, J commutes with $\eta = \frac{1}{2}(J^2 - E_3)$ and, hence, with $\frac{\partial \phi}{\partial \eta}$

The relation that connects the modified stress matrix T_a with the applied force-per-unit-mass matrix (replacing the relation $(\mathrm{div}_x\, T)^* + \rho_x F = 0$) is readily found In fact, the virtual work of all the forces

acting on any portion of the medium in any virtual deformation is furnished by the formula

$$\text{Virtual work} = \int_{V_x} \rho_x(\delta x)^* F \, dV_x + \int_{S_x} (\delta x)^* f \, dS_x$$

$$= \int_{V_a} \rho_a(\delta x)^* F \, dV_a + \int_{S_a} (\delta x)^* T_a \, dS^a$$

$$= \int_{V_a} \{\rho_a(\delta x)^* F + \text{div}_a \, (T_a{}^* \delta x)\} \, dV_a$$

where, if $p = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ is any 3×1 matrix whose elements are differentiable functions of (a, b, c), we understand by $\text{div}_a \, p$ the sum $p_a + q_b + r_c$. Similarly, if P is any $3 \times m$ matrix we understand by $\text{div}_a P$ the $1 \times m$ matrix whose elements are obtained by applying the operation div_a to the various columns of P.

This virtual work must be zero for the virtual translations (for which δx is a constant 3×1 matrix) for every portion V_a of the medium, and so

$$(\text{div}_a \, T_a{}^*)^* + \rho_a F = 0.$$

On substituting for T_a its value $J \dfrac{\partial \phi}{\partial \eta}$, so that $T_a{}^* = \left(\dfrac{\partial \phi}{\partial \eta}\right)^* J^*$, we obtain the basic equations of equilibrium of the medium in a form suitable for employing the initial coordinates (a, b, c), rather than the final coordinates (x, y, z), of a typical particle of the medium as independent variables:

$$\left[\text{div}_a \left\{\left(\frac{\partial \phi}{\partial \eta}\right)^* J^*\right\}\right]^* + \rho_a F = 0.$$

When the medium is isotropic $\left(\text{so that } \dfrac{\partial \phi}{\partial \eta} \text{ is symmetric}\right)$ and the applied mass force is either absent or negligible, these equations simplify to

$$\left\{\text{div}_a \frac{\partial \phi}{\partial \eta} J^*\right\}^* = 0,$$

or, equivalently,

$$\text{div}_a \left(\frac{\partial \phi}{\partial \eta} J^*\right) = 0.$$

Note 1. In the infinitesimal theory of elasticity, $\dfrac{\partial \phi}{\partial \eta}$ is replaced by

$\lambda I_1 + 2\mu\eta$ (the initial stress being zero), and the product of any element of $\dfrac{\partial\phi}{\partial\eta}$ by the derivative of any element of J^* with respect to a, b, or c is neglected The equation $\operatorname{div}_a\left(\dfrac{\partial\phi}{\partial\eta}J^*\right) = 0$ reduces, then, to $\left(\operatorname{div}_a \dfrac{\partial\phi}{\partial\eta}\right) J^* = 0$ or, equivalently (since J^* is non-singular), to

$$\operatorname{div}_a (\lambda I_1 + 2\mu\eta) = 0$$

If we assume the medium to be homogeneous, so that λ and μ are constant functions of (a, b, c), this equation yields

$$\lambda(I_1)_a + 2\mu \operatorname{div}_a \eta = 0$$

Note 2 In the derivation of the basic equation

$$(\operatorname{div}_a T_a^*)^* + \rho_a F = 0$$

we have assumed that the final reference frame is the same for every point of the medium (so that the virtual translations are characterized by the constancy of the 3×1 matrix δx) It sometimes happens (for example, when we wish to use space polar or cylindrical coordinates) that the directions of the axes of the final reference frame vary from point to point of the medium Let R_1 be the rotation matrix that transforms any convenient fixed reference frame to the (moving) final reference frame If $\delta\xi$ is the 3×1 matrix whose elements furnish the coordinates in the fixed reference frame of the vector whose coordinates in the moving final reference frame are furnished by the elements of the 3×1 matrix δx, we have $\delta x = R_1^*\delta\xi$, and the virtual translations are characterized by the constancy of the 3×1 matrix $\delta\xi$ The basic equation is now (show this)

$$\{\operatorname{div}_a (T_a^* R_1^*)\}^* + \rho_a R_1^* F = 0$$

When the applied mass or body forces are zero this reduces to

$$\operatorname{div}_a (T_a^* R_1^*) = 0$$

When we are using orthogonal curvilinear coordinates it is convenient to choose the fixed Cartesian reference frame to be that determined by the orthogonal curvilinear coordinates at the initial point (a, b, c) Then R_1 is the rotation matrix that rotates this reference frame into the reference frame determined by the orthogonal curvilinear coordinates at the final point (x, y, z) In performing the operation div_a on the 3×3 matrix $T_a^* R_1^*$, we must use the form of the divergence operator which is appropriate to the orthogonal curvilinear coordinates

we are using (see *Introduction to Applied Mathematics*, pp. 111–112[1]).

When (x, y, z) are linear functions of (a, b, c) the elements of $J = \frac{(x, y, z)}{(a, b, c)}$ are constants; hence the elements of $M = J^*J$ and of $\eta = \frac{1}{2}(M - E_3)$ are constants. Conversely, if the elements of η are constants (i.e., if the strain matrix is a constant matrix), the variables (x, y, z) are linear functions of the variables (a, b, c). In fact, the elements of M are constant and so $\Sigma x_a{}^2$ and $\Sigma x_a x_b$ are constants (Σ denoting summation with respect to x, y, and z). On differentiating the first of these with respect to a, b, and c, we obtain $\Sigma x_a x_{aa} = 0$, $\Sigma x_a x_{ab} = 0$, $\Sigma x_a x_{ac} = 0$. On differentiating the second with respect to a and using the relation $\Sigma x_a x_{ab} = 0$, we obtain $\Sigma x_b x_{aa} = 0$. Similarly, $\Sigma x_c x_{aa} = 0$, and the three relations $\Sigma x_a x_{aa} = 0$, $\Sigma x_b x_{aa} = 0$, $\Sigma x_c x_{aa} = 0$ assure us, since the matrix J^* is non-singular, that $x_{aa} = 0$, $y_{aa} = 0$, $z_{aa} = 0$. Similarly, $x_{bb} = 0$, $y_{bb} = 0$, $z_{bb} = 0$ and $x_{cc} = 0$, $y_{cc} = 0$, $z_{cc} = 0$. We have already seen that $\Sigma x_a x_{ab} = 0$, and, on interchanging a and b, this yields $\Sigma x_b x_{ab} = 0$. On differentiating the constant $\Sigma x_c x_a$ with respect to b, we obtain $\Sigma x_c x_{ab} = -\Sigma x_a x_{bc}$ and, on differentiating the constant $\Sigma x_b x_c$ with respect to a, we obtain $\Sigma x_c x_{ab} = -\Sigma x_b x_{ca}$. On differentiating the constant $\Sigma x_a x_b$ with respect to c, we obtain $\Sigma x_a x_{bc} + \Sigma x_b x_{ca} = 0$, and so $\Sigma x_c x_{ab} = 0$. The three relations $\Sigma x_a x_{ab} = 0$, $\Sigma x_b x_{ab} = 0$, $\Sigma x_c x_{ab} = 0$ assure us, since the matrix J^* is non-singular, that $x_{ab} = 0$, $y_{ab} = 0$, $z_{ab} = 0$. Similarly, $x_{bc} = 0$, $y_{bc} = 0$, $z_{bc} = 0$, and $x_{ca} = 0$, $y_{ca} = 0$, $z_{ca} = 0$. Hence all the second derivatives of (x, y, z) with respect to (a, b, c) are zero and so the variables (x, y, z) are linear functions of (a, b, c). We say that a strain is *homogeneous* when the strain matrix η is constant. We have, then, the following result:

A strain is homogeneous when, and only when, the Jacobian matrix $J = \frac{(x, y, z)}{(a, b, c)}$ *is a constant matrix. When this is so the variables* (x, y, z) *are linear functions of the variables* (a, b, c).

When the strain is homogeneous $\left(\frac{\partial \phi}{\partial \eta}\right)^* J^*$ is a constant matrix and so we have the following result:

When the strain is homogeneous the applied force per unit mass is zero (so that when the applied force per unit mass is not zero the strain cannot be homogeneous).

The following sections discuss two important examples of homogeneous strain.

[1] By Francis D. Murnaghan, John Wiley & Sons, 1948.

2. Simple shear

In *simple shear* the deformation is described by the equations

$$x = a + kb, \qquad y = b, \qquad z = c,$$

thus each particle is displaced, parallel to the x axis, an amount proportional to its distance from the b-plane We have seen (Example 1, p 33) that the principal axes of strain are obtained by rotating the (a, b) -axes around the c-axis through an angle $\alpha = \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\frac{k}{2}$. The characteristic numbers of M are $(1 + k \sin 2\alpha + k^2 \sin^2 \alpha,$ $1 - k \sin 2\alpha + k^2 \cos^2 \alpha, 1)$, and if we use the relation $k = -2 \cot 2\alpha$ these turn out to be $(\tan^2 \alpha, \cot^2 \alpha, 1)$ Hence the characteristic numbers of $M^{½}$ are $(\tan \alpha, \cot \alpha, 1)$, and so

$$M^{½} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tan\alpha & 0 & 0 \\ 0 & \cot\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin 2\alpha & -\cos 2\alpha & 0 \\ -\cos 2\alpha & 2\operatorname{cosec} 2\alpha - \sin 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (4 + k^2)^{-½} \begin{pmatrix} 2 & k & 0 \\ k & 2 + k^2 & 0 \\ 0 & 0 & (4 + k^2)^{½} \end{pmatrix}$$

Hence $M^{-½} = (4 + k^2)^{-½} \begin{pmatrix} 2 + k^2 & -k & 0 \\ -k & 2 & 0 \\ 0 & 0 & (4 + k^2)^{½} \end{pmatrix}$ and so the rotation matrix $R = JM^{-½}$ is given by the formula

$$R = (4 + k^2)^{-½} \begin{pmatrix} 2 & k & 0 \\ -k & 2 & 0 \\ 0 & 0 & (4 + k^2)^{½} \end{pmatrix} = \begin{pmatrix} \sin 2\alpha & -\cos 2\alpha & 0 \\ \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence R is a rotation through $\frac{\pi}{2} - 2\alpha = -\tan^{-1}\frac{k}{2}$ around the c axis Thus the coordinates of the stress tensor T, in the reference frame obtained by rotating the original (x, y, z)-axes around the z-axis through the angle $-\tan^{-1}\frac{k}{2}$, are the elements of the matrix

$$R^*TR = \left(\frac{\rho_x}{\rho_a}\right) M^{½} \frac{\partial\phi}{\partial\eta} M^{½}$$

The coordinates of T in the original (x, y, z)-reference frame are the elements of the matrix

$$T = \left(\frac{\rho_x}{\rho_a}\right) J \frac{\partial \phi}{\partial \eta} J^*.$$

Since $\det J = 1$, $\rho_x = \rho_a$ (so that no compression is involved or, equivalently, volumes are preserved in a simple shear). The modified stress matrix is $T_a = J \dfrac{\partial \phi}{\partial \eta}$.

We first consider the situation in which the medium is isotropic, and we take the initial stress to be a hydrostatic pressure p. Then

$$\phi = -pI_1 + \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + \frac{l + 2m}{3} I_1^3 - 2mI_1I_2 + nI_3,$$

$$\frac{\partial \phi}{\partial \eta} = -pE_3 + (\lambda I_1 E_3 + 2\mu\eta) + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n \operatorname{co} \eta$$

where

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{co} \eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}k^2 \end{pmatrix}.$$

Since we have stopped the development of $\phi(\eta)$ at terms of the third order in the elements of η, we are entitled only to keep terms not involving k to a higher power than the second in the expression $J \dfrac{\partial \phi}{\partial \eta}$. Since $I_1 = \frac{1}{2}k^2$, $I_2 = -\frac{1}{4}k^2$, we have, to this order of approximation,

$$\frac{\partial \phi}{\partial \eta} = \left\{-p + \frac{1}{2}(\lambda + m)k^2\right\} E_3 + 2\mu\eta + \eta \operatorname{co} \eta,$$

$$T_a = J \frac{\partial \phi}{\partial \eta} = -pE_3 + \begin{pmatrix} \frac{1}{2}(\lambda + 2\mu + m)k^2 & (\mu - p)k & 0 \\ \mu k & \frac{1}{2}(\lambda + 2\mu + m)k^2 & 0 \\ 0 & 0 & \frac{1}{2}(\lambda + m - \frac{1}{2}n)k^2 \end{pmatrix}.$$

The force on a matrix element of area dS^x of the boundary of the deformed medium is given by $T_a\, dS^a$. Since $(J^*)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have $dS^x = dS^a$, $dS^y = dS^b - k\, dS^a$, $dS^z = dS^c$. An element of area which lay initially in the a-plane (i.e., for which $dS^b = 0$, $dS^c = 0$) is deformed into an element of area $dS^a(1, -k, 0)$ whose magnitude is

the product of the magnitude of dS^a by $(1 + k^2)^{1/2}$ The force, per unit initial area, on an element of area which lay initially in, or parallel to, the a plane is found, then, by multiplying the first column of T_a by $(1 + k^2)^{-1/2} = 1 - \frac{1}{2}k^2$ Thus the coordinates of the force (per unit initial area) that acts upon an element of area which was initially in, or parallel to, the a-plane are $(-p + \frac{1}{2}(\lambda + 2\mu + m + p)k^2, \mu k, 0)$ Similarly, the coordinates of the force (per unit initial area) that acts upon an element of area which was originally in, or parallel to, the b-plane are $((\mu - p)k, -p + \frac{1}{2}(\lambda + 2\mu + m)k^2, 0)$, and the coordinates of the force (per unit initial area) that acts upon an element of area which was originally in or parallel to, the c-plane are $(0, 0, -p + \frac{1}{2}(\lambda + m - \frac{1}{2}n)k^2)$ If $p = 0$ the first-order part of this force per unit area is furnished by the matrix

$$k\begin{pmatrix} 0 & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is the result of the linear or infinitesimal theory of elasticity The *second-order correction* that must be added to this is furnished by the matrix

$$\frac{1}{2}k^2\begin{pmatrix} \lambda + 2\mu + m & 0 & 0 \\ 0 & \lambda + 2\mu + m & 0 \\ 0 & 0 & \lambda + m - \frac{1}{2}n \end{pmatrix}$$

$$= \frac{1}{2}(\lambda + 2\mu + m)k^2E_3 - (\mu + \tfrac{1}{4}n)k^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus in order to produce the simple shear a hydrostatic tension of magnitude $\frac{1}{2}(\lambda + 2\mu + m)k^2$ must be applied to the medium and, in addition, a simple linear compression (= negative tension) of magnitude $(\mu + \frac{1}{4}n)k^2$ parallel to the z-axis, these forces being added to the shearing force furnished by the linear theory If these surface forces which are necessary, in addition to the shear forces furnished by the linear theory, to maintain the simple shear are not supplied, the medium will contract (because of the absence of the hydrostatic tension), furthermore, it will expand in the direction of the z axis and contract in any direction perpendicular to this, because of the absence of the simple linear compression in the direction of the z axis (see the following section, in which simple tension is treated) Since both the hydrostatic pressure (= absence of hydrostatic tension) and the simple linear tension (= absence of simple linear compression) are second-order effects, the

magnitude of each having k^2 as a factor, we may use the linear theory in appraising the effects due to them. Thus the deformation due to the hydrostatic pressure is furnished by the formulas $x' = (1 - \beta)x$, $y' = (1 - \beta)y$, $z' = (1 - \beta)z$; $\beta = \dfrac{\lambda + 2\mu + m}{2(3\lambda + 2\mu)} k^2$. It will be shown in the next section that the deformation due to the simple linear tension is

$$x'' = \left\{1 - \sigma \frac{\mu + \frac{1}{4}n}{E} k^2\right\} x', \qquad y'' = \left\{1 - \sigma \frac{\mu + \frac{1}{4}n}{E} k^2\right\} y',$$
$$z'' = \left\{1 + \frac{\mu + \frac{1}{4}n}{E} k^2\right\} z'$$

where $\sigma = \dfrac{\lambda}{2(\lambda + \mu)}$ is Poisson's ratio and $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ is Young's modulus. On combining these deformations with the simple shear $x = a + kb$, $y = b$, $z = c$, we obtain (on neglecting powers of k higher than the second)

$$x'' = (1 - \alpha k^2)a + kb, \qquad y'' = (1 - \alpha k^2)b, \qquad z'' = (1 + \gamma k^2)c$$

where $\alpha = \dfrac{\lambda + 2\mu + m}{2(3\lambda + 2\mu)} + \dfrac{\sigma}{E}\left(\mu + \dfrac{1}{4}n\right)$, $\gamma = \dfrac{\mu + \frac{1}{4}n}{E} - \dfrac{\lambda + 2\mu + m}{2(3\lambda + 2\mu)}$.

In order to verify the correctness of this solution we have merely to start with the deformation

$$x = (1 - \alpha k^2)a + kb, \qquad y = (1 - \alpha k^2)b, \qquad z = (1 + \gamma k^2)c$$

where α and γ are undetermined constants. We obtain (always neglecting all powers of k higher than the second)

$$J = \begin{pmatrix} 1 - \alpha k^2 & k & 0 \\ 0 & 1 - \alpha k^2 & 0 \\ 0 & 0 & 1 + \gamma k^2 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 - 2\alpha k^2 & k & 0 \\ k & 1 + (1 - 2\alpha)k^2 & 0 \\ 0 & 0 & 1 + 2\gamma k^2 \end{pmatrix},$$

$$r = \begin{pmatrix} -\alpha k^2 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & (\frac{1}{2} - \alpha)k^2 & 0 \\ 0 & 0 & \gamma k^2 \end{pmatrix}, \quad I_1 = (\tfrac{1}{2} - 2\alpha + \gamma)k^2, \quad I_2 = -\tfrac{1}{4}k^2,$$

$$\frac{\partial\phi}{\partial\gamma} = \left\{\lambda\left(\frac{1}{2} - 2\alpha + \gamma\right) + \frac{1}{2}m\right\} k^2 E_3 +$$
$$\mu \begin{pmatrix} -2\alpha k^2 & k & 0 \\ k & (1 - 2\alpha)k^2 & 0 \\ 0 & 0 & 2\gamma k^2 \end{pmatrix} - \frac{1}{4}nk^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$J\frac{\partial\phi}{\partial\eta} = \begin{pmatrix} \{\lambda(\frac{1}{2} - 2\alpha + \gamma) + \frac{1}{2}m + \mu(1 - 2\alpha)\}k^2 & \mu k & 0 \\ \mu k & \{\lambda(\frac{1}{2} - 2\alpha + \gamma) + \frac{1}{2}m + \mu(1 - 2\alpha)\}k^2 & 0 \\ 0 & 0 & \{\lambda(\frac{1}{2} - 2\alpha + \gamma) + \frac{1}{2}m + 2\mu\gamma - \frac{1}{4}n\}k^2 \end{pmatrix}$$

In order that $J\dfrac{\partial\phi}{\partial\eta}$ should reduce to $\begin{pmatrix} 0 & \mu k & 0 \\ \mu k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, the undetermined constants α and γ must satisfy the two equations

$$2(\lambda + \mu)\alpha - \lambda\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \mu$$

$$2\lambda\alpha - (\lambda + 2\mu)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m - \tfrac{1}{4}n,$$

and a *simple calculation (perform it) shows that α and γ have the* values given above

The values of α and γ given above correspond to an applied force per unit area on the planes, into which planes parallel to the a-plane are deformed, of amount μk parallel to the y axis and an applied force per unit area on the planes, into which planes parallel to the b-plane are deformed, of amount μk parallel to the x-axis The second of these applied forces is a *shearing force* since the orientation of any plane parallel to the b-plane is not changed by the deformation (any such plane being deformed into a plane parallel to it), but the first is not Any plane element of area of the a plane is deformed into an element of area which is perpendicular to the vector $v(1 + (\gamma - \alpha)k^2, -k, 0)$, for, on setting a = constant in the equations defining the deformation, we obtain $dx = k\,db$, $dy = (1 - \alpha k^2)db$, $dz = (1 + \gamma k^2)dc$, and so $dS^x\ dS^y\ dS^z = 1 + (\gamma - \alpha)k^2\ \ -k\ \ 0$ (powers of k above the second being methodically neglected) If we wish, then, the forces on those planes, into which planes parallel to the a-plane are deformed, to be shearing forces, the first column vector of T_a *must be perpendicular to* the vector $v(1 + (\gamma - \alpha)k^2, -k, 0)$ We start out with the trial deformation

$$x = (1 - \alpha k^2)a + kb, \qquad y = (1 - \beta k^2)b, \qquad z = (1 + \gamma k^2)c$$

where α, β, and γ are undetermined constants We find, as before,

$$J = \begin{pmatrix} 1 - \alpha k^2 & k & 0 \\ 0 & 1 - \beta k^2 & 0 \\ 0 & 0 & 1 + \gamma k^2 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 - 2\alpha k^2 & k & 0 \\ k & 1 + (1 - 2\beta)k^2 & 0 \\ 0 & 0 & 1 + 2\gamma k^2 \end{pmatrix},$$

$$\eta = \begin{pmatrix} -\alpha k^2 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & (\frac{1}{2} - \beta)k^2 & 0 \\ 0 & 0 & \gamma k^2 \end{pmatrix}, I_1 = (\tfrac{1}{2} - \alpha - \beta + \gamma)k^2, I_2 = -\tfrac{1}{4}k^2,$$

$$\frac{\partial \phi}{\partial \eta} = \left\{\lambda\left(\frac{1}{2} - \alpha - \beta + \gamma\right) + \frac{1}{2}m\right\} k^2 E_3 + \mu\begin{pmatrix} -2\alpha k^2 & k & 0 \\ k & (1 - 2\beta)k^2 & 0 \\ 0 & 0 & 2\gamma k^2 \end{pmatrix} - \frac{1}{4}nk^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$J\frac{\partial \phi}{\partial \eta} = \begin{pmatrix} \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m + \mu(1 - 2\alpha)\}k^2 & \mu k & 0 \\ \mu k & \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m + \mu(1 - 2\beta)\}k^2 & 0 \\ 0 & 0 & \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m + 2\mu\gamma\}k^2 - \frac{1}{4}n \end{pmatrix}.$$

The three equations that serve to determine α, β, and γ are

$$(\lambda + 2\mu)\alpha + \lambda\beta - \lambda\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m,$$

$$\lambda\alpha + (\lambda + 2\mu)\beta - \lambda\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \mu,$$

$$\lambda\alpha + \lambda\beta - (\lambda + 2\mu)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m - \tfrac{1}{4}n.$$

From the first two of these we obtain $\beta = \alpha + \frac{1}{2}$, and, on substituting this value in the second and third, we obtain

$$2(\lambda + \mu)\alpha - \lambda\gamma = \tfrac{1}{2}m,$$

$$2\lambda\alpha - (\lambda + 2\mu)\gamma = \tfrac{1}{2}m - \tfrac{1}{4}n,$$

and so

$$\alpha = \frac{m}{2(3\lambda + 2\mu)} + \frac{n\lambda}{8\mu(3\lambda + 2\mu)}, \gamma = -\frac{m}{2(3\lambda + 2\mu)} + \frac{n(\lambda + \mu)}{4\mu(3\lambda + 2\mu)}.$$

With these values of α, β, γ, $T_a = J\dfrac{\partial \phi}{\partial \eta}$ is

$$T_a = \begin{pmatrix} \mu k^2 & \mu k & 0 \\ \mu k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the shearing force per unit area on planes into which planes parallel to the a-plane and to the b-plane are deformed is μk. The deformation is

$$x = (1 - \alpha k^2)a + kb, \qquad y = \{1 - (\alpha + \tfrac{1}{2})k^2\}b, \qquad z = (1 + \gamma k^2)c,$$

and the compression ratio $\dfrac{\rho_x}{\rho_a}$ is furnished by the formula

$$\frac{\rho_x}{\rho_a} = (\det J)^{-1} = 1 + \left(\frac{1}{2} + 2\alpha - \gamma\right)k^2 = 1 + \left\{\frac{1}{2} + \frac{6m - n}{4(3\lambda + 2\mu)}\right\}k^2.$$

The stress matrix T is

$$T = \left(\frac{\rho_x}{\rho_a}\right) J \frac{\partial \phi}{\partial \eta} J^* = \left(\frac{\rho_x}{\rho_a}\right) T_a J^* = \begin{pmatrix} 2\mu k^2 & \mu k & 0 \\ \mu k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If the initial hydrostatic pressure $p \neq 0$, the values of the elastic constants (λ, μ, l, m, n) are different since these are functions of p. The equations determining α, β, and γ are also slightly modified since $\dfrac{\partial \phi}{\partial \eta}$ and, hence, $T_a = J \dfrac{\partial \phi}{\partial \eta}$ involve p. We have (show this)

$$T_a = \begin{pmatrix} \begin{matrix} -p + \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m \\ + \mu(1 - 2\alpha) + p\alpha\}k^2 \end{matrix} & (\mu - p)k & 0 \\ \mu k & \begin{matrix} -p + \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m + \mu(1 - 2\beta) \\ + p\beta\}k^2 \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} -p + \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m + 2\mu\gamma \\ - \frac{1}{4}n - p\gamma\}k^2 \end{matrix} \end{pmatrix}$$

An element of area of magnitude dS_a of the a plane is transformed into the matrix element of area

$$dS^x = dS_a \begin{pmatrix} 1 + (\gamma - \beta)k^2 \\ -k \\ 0 \end{pmatrix}$$

whose magnitude is $dS_a \{1 + (\frac{1}{2} + \gamma - \beta)k^2\}$. Hence the resolved part of the force per unit area on dS^x, normal to it, is (show this) the product of

$$-p + \{\lambda(\tfrac{1}{2} - \alpha - \beta + \gamma) + \tfrac{1}{2}m - 2\alpha\mu + p(\alpha + \beta - \gamma)\}k^2$$

by $1 - (\frac{1}{2} + \gamma - \beta)k^2$, i.e., $-p + \{\lambda(\frac{1}{2} - \alpha - \beta + \gamma) + \frac{1}{2}m - 2\alpha\mu + p(\frac{1}{2} + \alpha)\}k^2$, and if this is to be $-p$ (so that the *additional* applied force on the matrix element of area dS^x is a shearing force) the undetermined constants α, β, γ must satisfy the relation

$$(\lambda + 2\mu - p)\alpha + \lambda\beta - \lambda\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \tfrac{1}{2}p$$

An element of area of magnitude dS_a of the b plane is transformed into the matrix element of area

$$dS^x = dS_a \begin{pmatrix} 0 \\ 1 + (\gamma - \alpha)k^2 \\ 0 \end{pmatrix}$$

and so the resolved part of the force per unit area on dS^x, normal to it, is (show this)

$$-p + \{\lambda(\tfrac{1}{2} - \alpha - \beta + \gamma) + \tfrac{1}{2}m + \mu(1 - 2\beta) + p(\beta + \gamma - \alpha)\}k^2$$

Hence, in order that the additional applied force on dS^z be a shearing force, the undetermined constants α, β, γ must satisfy the relation

$$(\lambda + p)\alpha + (\lambda + 2\mu - p)\beta - (\lambda + p)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \mu.$$

Similarly (show this), on considering an element of area of the c-plane, we obtain the relation

$$(\lambda + p)\alpha + (\lambda + p)\beta - (\lambda + 2\mu - p)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m - \tfrac{1}{4}n.$$

When α, β, γ are assigned the values that satisfy these equations, we have

$$T_a = \begin{pmatrix} -p + (\mu - \tfrac{1}{2}p)k^2 & (\mu - p)k & 0 \\ \mu k & -p + p(\alpha - \gamma)k^2 & 0 \\ 0 & 0 & -p + p(\alpha + \beta)k^2 \end{pmatrix},$$

and, since $\dfrac{\rho_x}{\rho_a} = 1 - (\gamma - \alpha - \beta)k^2$,

$$T = \left(\frac{\rho_x}{\rho_a}\right) T_a J^* = \begin{pmatrix} -p + \{2\mu + p(\gamma - \beta - \tfrac{3}{2})\}k^2 & (\mu - p)k & 0 \\ (\mu - p)k & -p & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

Thus the initial pressure affects even the first-order approximation, the effect being an apparent reduction of the *modulus of rigidity* μ by p. Since, however, μ is itself a function of p all we can say is that $\mu - p$ plays, when $p \neq 0$, the role played by μ_0 when $p = 0$, μ_0 being the value of μ when $p = 0$.

On adding together the second and third of the three equations that serve to determine α, β, and γ, we obtain

$$(\lambda + p)\alpha + (\lambda + \mu)(\beta - \gamma) = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \tfrac{1}{2}\mu - \tfrac{1}{8}n,$$

and, on combining this with the first of the three equations, we obtain

$$\beta - \gamma = \frac{(\lambda + m)(\mu - p) + (\tfrac{1}{2}\mu - \tfrac{1}{8}n)(\lambda + 2\mu - p) - \tfrac{1}{2}p(\lambda + p)}{3\lambda\mu + 2\mu^2 - p(2\lambda + \mu)},$$

$$\alpha = \frac{\tfrac{1}{2}m\mu + \tfrac{1}{8}n\mu + \tfrac{1}{2}p(\lambda + \mu)}{3\lambda\mu + 2\mu^2 - p(2\lambda + \mu)}.$$

On subtracting the third of our three equations from the second, we obtain

$$\beta + \gamma = \frac{\tfrac{1}{2}\mu + \tfrac{1}{8}n}{\mu - p},$$

and these formulas determine α, β, and γ, when λ, μ, m, and n are known as functions of p Conversely, a knowledge (obtained by experiment) of how the compression ratio, for example, varies with p furnishes information as to the nature of the dependence on p of the function $(\beta - \gamma) + \alpha$ of λ, μ, m, n, and p

The theory is more complicated when the medium that is being subjected to the simple shear is, like wood, non-isotropic We consider first a medium that is elastically insensitive to any rotation around the a-axis (the direction of the shear) There are, then, three *additional second-order elastic constants and seven additional third* order elastic constants (the medium having fifteen elastic constants in all, five of the second order and ten of the third order) However, since I_1 and many of the elements of η are infinitesimals of the second order and not, as is generally the situation, of the first order, not all these ten additional elastic constants will affect our second order approximation Adopting the notation of Exercise 12, p 95, where the three additional second order elastic constants are denoted by $(\delta_1, \delta_2, \delta_3)$ and the seven additional third order elastic constants by $(\delta_4, \quad, \delta_{10})$, the addition to the matrix $\dfrac{\partial\phi}{\partial\eta}$ due to the non-isotropy of the medium is

$$\begin{pmatrix} \delta_1\eta_1 + \delta_2(I_1 + \eta_1) & 0 & 0 \\ 0 & \delta_2\eta_1 + \delta_3\eta_3 & 0 \\ 0 & 0 & \delta_2\eta_1 + \delta_3\eta_2 \end{pmatrix} =$$
$$k^2\begin{pmatrix} -\delta_1\alpha + \delta_2(\frac{1}{2} - 2\alpha - \beta + \gamma) & 0 & 0 \\ 0 & -\delta_2\alpha + \delta_3\gamma & 0 \\ 0 & 0 & -\delta_2\alpha + \delta_3(\frac{1}{2} - \beta) \end{pmatrix}$$

(Remember that, when evaluating $\dfrac{\partial\phi}{\partial\eta}$, ϕ must be written *symmetrically* and so $\eta_2\eta_3 - \eta_4{}^2$, for example must be replaced by $\eta_{bb}\eta_{cc} - \eta_{bc}\eta_{cb}$) Thus the seven additional third-order elastic constants do not affect our second order approximation (only the three additional second-order constants $(\delta_1, \delta_2, \delta_3)$ being involved) The addition to T_a is the same as the addition to $\dfrac{\partial\phi}{\partial\eta}$ (since the addition to $\dfrac{\partial\phi}{\partial\eta}$ is of the second order in k), and the addition to the stress matrix T is also, for the same reason, the same as the addition to $\dfrac{\partial\phi}{\partial\eta}$ The equations that serve to determine the constants α, β, and γ are (the initial stress being taken

to be zero)

$$(\lambda + 2\mu + \delta_1 + 2\delta_2)\alpha + (\lambda + \delta_2)\beta - (\lambda + \delta_2)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \tfrac{1}{2}\delta_2,$$
$$(\lambda + \delta_2)\alpha + (\lambda + 2\mu)\beta - (\lambda + \delta_3)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m + \mu,$$
$$(\lambda + \delta_2)\alpha + (\lambda + \delta_3)\beta - (\lambda + 2\mu)\gamma = \tfrac{1}{2}\lambda + \tfrac{1}{2}m - \tfrac{1}{4}n + \tfrac{1}{2}\delta_3.$$

When the values of α, β, and γ which satisfy these equations are used,

$$T_a = \begin{pmatrix} \mu k^2 & \mu k & 0 \\ \mu k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 2\mu k^2 & \mu k & 0 \\ \mu k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\frac{\rho_x}{\rho_a} = 1 + (\alpha + \beta - \gamma)k^2.$$

When the medium is insensitive to any rotation around the b-axis, the additional terms in ϕ are of the form

$$\begin{aligned} &\tfrac{1}{2}\delta_1\eta_2^2 + \delta_2 I_1\eta_2 + \delta_3(\eta_3\eta_1 - \eta_5^2) + \tfrac{1}{3}\delta_4\eta_2^3 + \delta_5 I_1\eta_2^2 \\ &\quad + \delta_6\eta_2(\eta_3\eta_1 - \eta_5^2) + \delta_7 I_1(\eta_3\eta_1 - \eta_5^2) + \delta_8\eta_2(\eta_6^2 + \eta_4^2) \\ &\qquad + \delta_9 I_1(\eta_6^2 + \eta_4^2) + \delta_{10}\{\eta_4\eta_6(\eta_3 - \eta_1) + \eta_5(\eta_6^2 - \eta_4^2)\}, \end{aligned}$$

and so the additional terms in $\dfrac{\partial\phi}{\partial\eta}$ are

$$\begin{pmatrix} \delta_2\eta_2 + \delta_3\eta_3 + \delta_9\eta_6^2 & 0 & \tfrac{1}{2}\delta_{10}\eta_6^2 \\ 0 & \begin{matrix}\delta_1\eta_2 + \delta_2(I_1 + \eta_2) + \delta_8\eta_6^2 \\ + \delta_9\eta_6^2\end{matrix} & 0 \\ \tfrac{1}{2}\delta_{10}\eta_6^2 & 0 & \delta_2\eta_2 + \delta_3\eta_1 + \delta_9\eta_6^2 \end{pmatrix}$$
$$= k^2\begin{pmatrix} \delta_2(\tfrac{1}{2} - \beta) + \delta_3\gamma + \tfrac{1}{4}\delta_9 & 0 & \tfrac{1}{8}\delta_{10} \\ 0 & \begin{matrix}\delta_1(\tfrac{1}{2} - \beta) + \delta_2(1 - \alpha - 2\beta + \gamma) \\ + \tfrac{1}{4}\delta_8 + \tfrac{1}{4}\delta_9\end{matrix} & 0 \\ \tfrac{1}{8}\delta_{10} & 0 & \delta_2(\tfrac{1}{2} - \beta) - \delta_3\alpha + \tfrac{1}{4}\delta_9 \end{pmatrix}.$$

The additional third-order elastic constants δ_8, δ_9, and δ_{10} are now involved in our second-order approximation. Furthermore, there is now a shearing force on planes into which planes parallel to the c-plane are deformed (this shearing force being $\tfrac{1}{8}\delta_{10}k^2$ per unit area parallel to the x-axis).

If the medium is elastically insensitive to any rotation around the z-axis, the additional terms in $\dfrac{\partial\phi}{\partial\eta}$ (over and above the terms that would be present if the medium were isotropic), are furnished by the matrix (show this)

$$\begin{pmatrix} \delta_2\eta_3 + \delta_3\eta_2 & -\delta_3\eta_6 & 0 \\ -\delta_3\eta_6 & \delta_2\eta_3 + \delta_3\eta_1 & 0 \\ 0 & 0 & \delta_1\eta_3 + \delta_2(I_1 + \eta_3) \end{pmatrix} = -k\begin{pmatrix} 0 & \delta_3 & 0 \\ \delta_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ k^2\begin{pmatrix} \delta_2\gamma + \delta_3(\frac{1}{2} - \beta) & 0 & 0 \\ 0 & \delta_2\gamma - \delta_3\alpha & 0 \\ 0 & 0 & \delta_1\gamma + \delta_2(\frac{1}{2} - \alpha - \beta + 2\gamma) \end{pmatrix}$$

Thus the presence of the additional second-order elastic constant δ_3 affects the *first-order* or *linear* approximation, the effect being an apparent reduction of the *modulus of rigidity* μ by δ_3 We leave to the reader the task of writing out the equations that serve to determine the constants α, β, and γ in this, and the preceding, case and also the task of developing the appropriate theory when the medium is under an initial hydrostatic pressure

3. Simple tension; the linear theory

The problem of simple tension is that of the deformation of a homogeneous cylinder under the application of equal and opposite forces applied uniformly over the two ends of the cylinder We take the c-axis parallel to the generators of the cylinder and consider first a cylinder that is elastically isotropic and initially unstressed We start off with the trial deformation

$$x = (1 - \sigma k)a, \qquad y = (1 - \sigma k)b, \qquad z = (1 + k)c$$

where k is an undetermined constant which is so small that we neglect any power of it higher than the first σ is a second undetermined constant that *is, as we shall see a function of the elastic constants* of the medium (being independent of the applied force) Our trial deformation is such that J, and hence η, are diagonal, the three diagonal elements of η are

$$\eta_1 = -\sigma k, \qquad \eta_2 = -\sigma k, \qquad \eta_3 = k$$

and so $I_1 = (1 - 2\sigma)k \quad \dfrac{\partial\phi}{\partial\eta} = \lambda I_1 E_3 + 2\mu\eta$ is diagonal its first and second diagonal elements having the common value $\{\lambda(1 - 2\sigma) - 2\mu\sigma\}k$ Since we are neglecting powers of k higher than the first, the matrix $T_a = J\dfrac{\partial\phi}{\partial\eta}$ is the same as the matrix $\dfrac{\partial\phi}{\partial\eta}$, and, since there is no applied force on the sides of the cylinder, σ must be such that the first (and second) diagonal elements of T_a are zero (for the matrix element of area has its third coordinate zero on the sides of the cylinder and thus at least one of its first two coordinates is different from zero)

Hence

$$\sigma = \frac{\lambda}{2(\lambda + \mu)};$$

σ is known as *Poisson's ratio;* for many media its value is about 0.3. If l is the (initial) length of the cylinder the two ends of the cylinder *may be taken to be given, before the deformation takes place,* by the equations $c = 0$, $c = l$. If the magnitude of the applied force per unit (initial) area on the deformed position of the end $c = l$ is denoted by P, we have (since the third diagonal element of T_a is $\lambda I_1 + 2\mu\eta_3 = \{\lambda(1 - 2\sigma) + 2\mu\}k = 2\mu(1 + \sigma)k$), $2\mu(1 + \sigma)k = P$ and so $k = \dfrac{P}{2\mu(1 + \sigma)}$. We denote $2\mu(1 + \sigma)$ by E and we term E the *Young's modulus* of the medium:

$$E = 2\mu(1 + \sigma) = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

Then $k = \dfrac{P}{E}$. Thus our assumption that k is so small that its square may be neglected means that P is so small in comparison with E that the square of $\dfrac{P}{E}$ may be neglected. Since $dx\,dy = (1 - \sigma k)^2\,da\,db = (1 - 2\sigma k)\,da\,db$, any element of area of a cross section of the cylinder is decreased in magnitude by the factor $1 - 2\sigma k$; if, then, we denote by P' the magnitude of the stress on the end of the cylinder (i.e., the magnitude of the applied force per unit final area), we have $P'(1 - 2\sigma k) = P$, and so

$$k = \frac{P}{E} = \frac{P'(1 - 2\sigma k)}{E}.$$

On multiplying both sides of this equation by $1 + 2\sigma k$ and neglecting k^2, we obtain

$$k = \frac{P'}{E}.$$

In words, to the degree of approximation furnished by the linear theory it makes no difference whether P denotes the applied force per unit initial area or the applied force per unit final area. In either event $k = \dfrac{P}{E}$, and the deformation is furnished by the formulas

$$x = \left(1 - \sigma\frac{P}{E}\right)a, \qquad y = \left(1 - \sigma\frac{P}{E}\right)b, \qquad z = \left(1 + \frac{P}{E}\right)c.$$

The relative extension $\dfrac{(z-c)}{c}$ is equal to $\dfrac{P}{E}$ The graph of P, as a function of the relative extension, is a straight line through the origin having slope E

If the initial stress, instead of being zero, is a hydrostatic pressure p, the first diagonal element of $\dfrac{\partial\phi}{\partial\eta}$ is $-p + \{\lambda(1-2\sigma) - 2\mu\sigma\}k$ and so the first diagonal element of $T_a = J\dfrac{\partial\phi}{\partial\eta}$ is $-p + \{\lambda(1-2\sigma) - 2\mu\sigma + p\sigma\}k$ An element of area which was originally normal to the a axis remains normal to the a-axis after the deformation but its magnitude is increased by the factor $1 + k(1-\sigma)$ The applied force on such an element being a pressure p, we must have $-p + \{\lambda(1-2\sigma) - 2\mu\sigma + p\sigma\}k = -p\{1 + k(1-\sigma)\}$ and so $\lambda(1-2\sigma) - 2\mu\sigma + p = 0$ Hence

$$\sigma = \frac{\lambda + p}{2(\lambda + \mu)}$$

Thus the effect of the initial pressure is to change the formula furnishing Poisson's ratio to the extent of replacing the numerator of the fraction by $\lambda + p$ Of course, λ and μ are themselves functions of p so that σ depends on p not only explicitly (through the term p in the numerator $\lambda + p$ of the fraction) but also implicitly through λ and μ

The third diagonal element of $\dfrac{\partial\phi}{\partial\eta}$ is $-p + \{\lambda(1-2\sigma) + 2\mu\}k$, and so the third diagonal element of T_a is $-p + \{\lambda(1-2\sigma) + 2\mu - p\}k$ An element of area which was originally normal to the c-axis remains normal to the c-axis but its magnitude is decreased by the factor $1 - 2\sigma k$ If, then, the applied force is of magnitude P per unit initial area (in addition to a pressure p), we must have

$$-p + \{\lambda(1-2\sigma) + 2\mu - p\}k = P - p(1 - 2\sigma k),$$

and so

$$\{\lambda(1-2\sigma) + 2\mu - p - 2p\sigma\}k = P$$

or, equivalently,

$$k = \frac{P}{E} \text{ where } E = \lambda(1-2\sigma) + 2\mu - p - 2p\sigma$$

As when $p = 0$, it does not matter whether P denotes the magnitude of the applied force per unit initial area or the magnitude of the applied force per unit final area (since we are neglecting k^2) Since $\lambda(1 -$

$2\sigma) = 2\mu\sigma - p$ we have

$$E = 2(\mu - p)(1 + \sigma) = \frac{(\mu - p)(3\lambda + 2\mu + p)}{\lambda + \mu}.$$

Thus the effect of the initial pressure, in the formulas furnishing σ and E, is to increase λ by p and to decrease μ by p (so that the denominators of the fractions that furnish σ and E are unaffected formally; they are, of course, affected by the fact that $\lambda + \mu$ is a function of p).

If the initial stress is a non-scalar stress, we know that the medium cannot be elastically isotropic (why?). Let us consider an initial stress matrix all of whose elements are zero save the one in the third row and third column, and let us denote this non-zero element by P. We assume, further, that the medium is elastically insensitive to any rotation around the c-axis. Then $\phi(\eta)$ is of the form

$$\phi(\eta) = P\eta_3 + \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + \frac{1}{2}\delta_1\eta_3^2 + \delta_2 I_1\eta_3 + \delta_3(\eta_1\eta_2 - \eta_6^2)$$

and so, since η is, by hypothesis, a diagonal matrix, $\dfrac{\partial\phi}{\partial\eta}$ is a diagonal matrix whose diagonal elements are

$$\lambda I_1 + 2\mu\eta_1 + \delta_2\eta_3 + \delta_3\eta_2,\ \lambda I_1 + 2\mu\eta_2 + \delta_2\eta_3 + \delta_3\eta_1,\ P + \lambda I_1 + 2\mu\eta_3 + \delta_1\eta_3 + \delta_2(I_1 + \eta_3),$$

i.e.,

$$\{\lambda(1 - 2\sigma) - 2\mu\sigma + \delta_2 - \delta_3\sigma\}k,\ \{\lambda(1 - 2\sigma) - 2\mu\sigma + \delta_2 - \delta_3\sigma\}k,\ P + \{\lambda(1 - 2\sigma) + 2\mu + \delta_1 + 2\delta_2(1 - \sigma)\}k.$$

Hence, since the first and second diagonal elements of $T_a = J\dfrac{\partial\phi}{\partial\eta}$ must be zero (why?), σ is determined by the formula

$$\sigma = \frac{\lambda + \delta_2}{2\lambda + 2\mu + \delta_3}.$$

The equation determining k is

$$\{\lambda(1 - 2\sigma) + 2\mu + \delta_1 + 2\delta_2(1 - \sigma) + P\}k = \Delta P$$

where $P + \Delta P$ is the applied force per unit initial area. The coefficient of k is Young's modulus E and, since $\lambda(1 - 2\sigma) = 2\mu\sigma - \delta_2 + \delta_3\sigma$, we have

$$E = 2\mu(1 + \sigma) + \delta_1 + \delta_2(1 - 2\sigma) + \delta_3\sigma + P.$$

Since λ is the relative extension $\dfrac{(z-c)}{c}$, we have

$$\frac{z-c}{c} = \frac{\Delta P}{E}$$

where E is a function of P both explicitly and through the elastic constants μ, σ, δ_1, δ_2, and δ_3 On denoting $z - c$ by Δc, we obtain, on letting $\Delta P \to 0$, from the approximate equation $\dfrac{\Delta c}{c} = \dfrac{\Delta P}{E}$ the exact equation

$$\frac{dc}{c} = \frac{dP}{E},$$

and, on integrating this,

$$\log\left(1 + \frac{\Delta l}{l}\right) = \int_0^{P'} \frac{dP}{E}$$

where l is the length of the cylinder when there is no applied force and Δl is the extension when the applied force per unit final area is P' *Note* P' is the applied force per unit final area, and not the applied force per unit initial area, since P is the applied force per unit area, the area being evaluated when the cylinder is under the tension P

In order to perform the integration $\displaystyle\int_0^{P'} \frac{dP}{E}$, we must know how the Young's modulus E of the medium varies with P If we make the drastic assumption that μ and σ are independent of P and neglect the lack of isotropy introduced by the application of the tension P, E is the linear function $2\mu(1 + \sigma) + P$ of P and we obtain

$$\log\left(1 + \frac{\Delta l}{l}\right) = \log\left(1 + \frac{P'}{E_0}\right)$$

or, equivalently,

$$\frac{\Delta l}{l} = \frac{P'}{E_0}$$

where $E_0 = 2\mu(1 + \sigma)$ is the Young's modulus of the medium when unstressed Thus, under these unrealistic hypotheses, the extended linear theory furnishes the same result as the simple linear theory, namely

The relative extension is proportional to the applied stress (or force per unit final are

This result inal *ke's law* in its simplest form The formula $7 =$

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$\left(\frac{\rho_1}{\rho_3}\right) J \frac{\partial \phi}{\partial \eta} J^*$ may be regarded as the general form of statement of Hooke's law.

If we suppose that E may be sufficiently approximated, over the interval $(0, P')$, by the linear function $E_0 + \alpha P$, where α is a constant that is determined by experiment, we obtain

$$\log\left(1 + \frac{\Delta l}{l}\right) = \frac{1}{\alpha} \log\left(1 + \alpha \frac{P'}{E_0}\right)$$

or, equivalently,

$$\frac{\Delta l}{l} = \left(1 + \alpha \frac{P'}{E_0}\right)^{\frac{1}{\alpha}} - 1.$$

If α happens to be negative this formula indicates that $\Delta l \to \infty$ as $P' \to \frac{E_0}{-\alpha}$. In fact, on setting $\alpha = -\beta$, we obtain

$$\frac{\Delta l}{l} = \frac{1}{\left(1 - \beta \frac{P'}{E_0}\right)^{\frac{1}{\beta}}} - 1,$$

and, as $P' \to \frac{E_0}{\beta}$, $\left(1 - \beta \frac{P'}{E_0}\right)^{\frac{1}{\beta}} \to 0$. Thus the extended linear theory is capable of explaining the phenomenon of flow of a cylinder under tension; but in order for it to do so one must take into account the dependence of the elastic constants on the applied tension. It seems hardly likely that the mere dependence of μ and σ on P would be sufficient to account for this phenomenon of flow; it is to be expected that the elastic constants δ_1, δ_2, δ_3 (which measure, by their differences from zero, the lack of isotropy of the medium) must be considered.

4. Simple tension; the second-order approximation

We consider first a medium, supposed to be initially unstressed, that is isotropic. We assume the trial deformation $x = (1 - \sigma k + \beta k^2)a$, $y = (1 - \sigma k + \beta k^2)b$, $z = (1 + k + \delta k^2)c$ where $k = \frac{P}{E}$ and where σ, E are the elastic constants (Poisson's ratio and Young's modulus) of the simple linear theory; β and δ are two new undetermined constants. We neglect methodically powers of k above the second. η is a diagonal matrix whose diagonal elements are

$$\eta_1 = -\sigma k + (\beta + \tfrac{1}{2}\sigma^2)k^2 = \eta_2$$

$$\eta_3 = k + (\delta + \tfrac{1}{2})k^2$$

and so

$$I_1 = (1 - 2\sigma)k + (2\beta + \delta + \tfrac{1}{2} + \sigma^2)k^2$$
$$I_2 = (\sigma^2 - 2\sigma)k^2$$

The constant σ is such that the coefficient of k in the first diagonal element of $\lambda I_1 E_3 + 2\mu\eta$ is zero and so the first diagonal element of $\dfrac{\partial\phi}{\partial\eta}$ is the product of k^2 by $\lambda(2\beta + \delta + \frac{1}{2} + \sigma^2) + 2\mu(\beta + \frac{1}{2}\sigma^2) + l(1 - 2\sigma)^2 + 2m\sigma(1 + \sigma) - n\sigma$ Since we are neglecting powers of k above the second, this expression also furnishes the first diagonal element of T_a (why?) and so the two undetermined constants β and δ must satisfy the following relation (why?)

$$2(\lambda + \mu)\beta + \lambda\delta = -\lambda(\tfrac{1}{2} + \sigma^2) - \mu\sigma^2 - l(1 - 2\sigma)^2 - 2m\sigma(1 + \sigma) + n\sigma$$

The constant E is such that the linear part of the third diagonal element of T_a is P, the applied force per unit initial area Since the third diagonal element of $\dfrac{\partial\phi}{\partial\eta}$ is $\{\lambda(1 - 2\sigma) + 2\mu\}k + \{\lambda(2\beta + \delta + \frac{1}{2} + \sigma^2) + \mu(2\delta + 1) + l(1 - 2\sigma)^2 + 2m(1 - \sigma^2) + n\sigma^2\}k^2$, the third diagonal element of $T_a = J\dfrac{\partial\phi}{\partial\eta}$ is $P + \{\lambda(2\beta + \delta + \frac{3}{2} - 2\sigma + \sigma^2) + \mu(2\delta + 3) + l(1 - 2\sigma)^2 + 2m(1 - \sigma^2) + n\sigma^2\}k^2$ On setting the coefficient of k^2 in this expression equal to zero, we see that the undetermined constants β and δ must satisfy the relation

$$2\lambda\beta + (\lambda + 2\mu)\delta = -\lambda(\tfrac{3}{2} - 2\sigma + \sigma^2) - 3\mu - l(1 - 2\sigma)^2 - 2m(1 - \sigma^2) - n\sigma^2,$$

and, on solving for β and δ the two linear equations that they satisfy, we find

$$\beta = \frac{\lambda(3\lambda + 4\mu)}{8(\lambda + \mu)^2} - \frac{\mu^2 l}{(3\lambda + 2\mu)(\lambda + \mu)^2} + \frac{\lambda(3\lambda^2 + 6\lambda\mu + 4\mu^2)}{8\mu(3\lambda + 2\mu)(\lambda + \mu)^2}n$$

$$\delta = -\frac{3}{2} - \frac{\mu^2 l}{(3\lambda + 2\mu)(\lambda + \mu)^2} - \frac{(3\lambda + 2\mu)}{2(\lambda + \mu)^2}m - \frac{3\lambda^2}{4\mu(3\lambda + 2\mu)(\lambda + \mu)}n$$

The graph of the relative extension $\dfrac{(z - c)}{c}$ plotted against $k = \dfrac{P}{E}$ is a parabola passing through the origin with slope 1 and second derivative 2δ

The dependence of Young's modulus E on the applied tension, according to the second-order approximation, is obtained as follows: On writing the relative extension $\frac{(z-c)}{c}$ in the form $\frac{P}{E'}$, we have

$$\frac{P}{E'} = k + \delta k^2 = \frac{P}{E} + \delta\left(\frac{P}{E}\right)^2$$

and so

$$\frac{1}{E'} = \frac{1}{E}\left(1 + \delta\frac{P}{E}\right)$$

or, equivalently,

$$E' = E\left(1 - \delta\frac{P}{E}\right) = E - \delta P.$$

Thus the constant α of the extended linear theory is $-\delta$ if the second-order approximation is adopted. In order that the second-order approximation may furnish (through the method of the extended linear theory) an explanation of the phenomenon of flow, it is necessary that δ be positive; this being the case, the medium will flow before the applied force per unit area attains the value $\frac{E_0}{\delta}$.

To examine the effect of an initial hydrostatic pressure p (following the theory of the second-order approximation), we observe that the first diagonal element of $\frac{\partial\phi}{\partial\eta}$ is

$$\begin{aligned}-p + \{\lambda(1 - 2\sigma) - 2\mu\sigma\}k + \{\lambda(2\beta + \delta + \tfrac{1}{2} + \sigma^2) + \mu(2\beta \\ + \sigma^2) + l(1 - 2\sigma)^2 + 2m\sigma(1 + \sigma) - n\sigma\}k^2,\end{aligned}$$

and so the first diagonal element of $T_c = J\frac{\partial\phi}{\partial\eta}$ is

the coefficients of k^2 we obtain the equation

$$2(\lambda + \mu)\beta + (\lambda + p)\delta = -\lambda(\tfrac{1}{2} - \sigma + 3\sigma^2) - 3\mu\sigma^2 - l(1 - 2\sigma)^2 - 2m\sigma(1 + \sigma) + n\sigma + p\sigma$$

connecting the undetermined constants β and δ On using the relation $p = 2(\lambda + \mu)\sigma - \lambda$ we may write the right hand side of this equation in the form $-\lambda(\tfrac{1}{2} + \sigma)^2 - \mu\sigma^2 - l(1 - 2\sigma)^2 - 2m\sigma(1 + \sigma) + n\sigma$ To obtain a second equation connecting β and δ we observe that the third diagonal element of $\dfrac{\partial \phi}{\partial \eta}$ is

$$-p + \{\lambda(1 - 2\sigma) + 2\mu\}k + \{\lambda(2\beta + \delta + \tfrac{1}{2} + \sigma^2) + \mu(2\delta + 1) + l(1 - 2\sigma)^2 + 2m(1 - \sigma^2) + n\sigma^2\}k^2$$

The third diagonal element of T_a is the product of this expression by $1 + k + \delta k^2$ i e

$$-p + \{\lambda(1 - 2\sigma) + 2\mu - p\}k + \{\lambda(2\beta + \delta + \tfrac{3}{2} - 2\sigma + \sigma^2) + \mu(2\delta + 3) + l(1 - 2\sigma)^2 + 2m(1 - \sigma^2) + n\sigma^2 - p\sigma\}k^2$$

and this must be equal to the sum of P (the applied force per unit initial area) and the product of $-p$ by the ratio of $dx\,dy$ to $da\,db$ namely $1 - 2\sigma k + (2\beta + \sigma^2)k^2$ The Young s modulus E is deter mined (in accordance with the linear theory where the initial pressure is different from zero) so that the linear parts of the two sides of the equation obtained in this way are equal Equating the coefficients of k^2 we obtain the equation

$$2(\lambda + p)\beta + (\lambda + 2\mu - p)\delta = -\lambda(\tfrac{3}{2} - 2\sigma + \sigma^2) - 3\mu - l(1 - 2\sigma)^2 - 2m(1 - \sigma^2) - n\sigma^2 - p\sigma^2$$

connecting the undetermined constants β and δ The determinant of the two linear equations that β and δ must satisfy is

$$2(\mu - p)(3\lambda + 2\mu + p)$$

and so β and δ are determined unambiguously by the two equations unless $p - \mu$ The theory breaks down when $p = \mu$ because the effective rigidity of the medium is $\mu - p$ and so the smaller is $\mu - p$ the more liquid like (i e less able to support tension) is the medium *It is important to remember that μ is a function of p it may well be* that $\mu - p$ is for positive values of p, an increasing function of p and so $\mu - p$ is never zero

7

PARTICULAR PROBLEMS

1. The compression of a spherical shell

We consider an elastically isotropic spherical shell whose internal and external radii, when the shell is free from stress, are R_i and R_e, respectively. The problem we wish to solve is the following: What is the deformation of the shell under an applied internal pressure p_i and an applied external pressure p_e? From reasons of symmetry each particle of the shell undergoes a radial displacement whose magnitude is a function of the initial distance r of the particle from the center of the shell. If we denote this displacement by ku, the space polar coordinates (relative to a reference frame whose origin is at the center of the shell) of the final position of a particle whose initial coordinates had the space polar coordinates (r, θ, ϕ) are $(r + ku, \theta, \phi)$. Here k is a constant which is such that in the linear theory we neglect powers of k above the first whereas in the second-order approximation we neglect powers of k above the second. The space polar coordinates of the vector $\mathbf{da}$ are $(dr, r\,d\theta, r \sin\theta\, d\phi)$, and the space polar coordinates of the vector $\mathbf{dx}$ are $(dr + k\,du, (r + ku)\,d\theta, (r + ku)\sin\theta\,d\phi)$. Thus da is the 3×1 matrix

$$\begin{pmatrix} dr \\ r\,d\theta \\ r\sin\theta\,d\phi \end{pmatrix}$$

and dx is the 3×1 matrix

$$\begin{pmatrix} dr + k\,du \\ (r + ku)\,d\theta \\ (r + ku)\sin\theta\,d\phi \end{pmatrix}.$$

Hence the 3×3 matrix J that transforms, by means of the formula $dx = J\,da$, da into dx is the diagonal matrix

$$J = \begin{pmatrix} 1 + ku' & 0 & 0 \\ 0 & 1 + k\dfrac{u}{r} & 0 \\ 0 & 0 & 1 + k\dfrac{u}{r} \end{pmatrix}$$

where the prime indicates differentiation with respect to r It follows that the space polar coordinates of the strain matrix η are furnished by the elements of the 3×3 diagonal matrix

$$k\begin{pmatrix} u' & 0 & 0 \\ 0 & \frac{u}{r} & 0 \\ 0 & 0 & \frac{u}{r} \end{pmatrix} + \frac{1}{2}k^2\begin{pmatrix} (u')^2 & 0 & 0 \\ 0 & \frac{u^2}{r^2} & 0 \\ 0 & 0 & \frac{u^2}{r^2} \end{pmatrix}$$

Neglecting powers of k above the second we have

$$I_1 = k\left(u' + 2\frac{u}{r}\right) + \frac{1}{2}k^2\left\{(u')^2 + 2\frac{u^2}{r^2}\right\},$$

$$I_2 = k^2\left(\frac{u^2}{r^2} + \frac{2uu'}{r}\right)$$

$$\text{co } \eta = k^2\begin{pmatrix} \frac{u^2}{r^2} & 0 & 0 \\ 0 & \frac{uu'}{r} & 0 \\ 0 & 0 & \frac{uu'}{r} \end{pmatrix}$$

Since the medium is by hypothesis isotropic and free from initial stress

$$\frac{\partial\phi}{\partial\eta} = \lambda I_1 E_3 + 2\mu\eta + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta + n \text{ co } \eta$$

(In this equation, ϕ is, of course the energy density and not the third space polar coordinate) Thus $\frac{\partial\phi}{\partial\eta}$ is diagonal and so $T_a = J\frac{\partial\phi}{\partial\eta}$ is diagonal We assume that the shell is free from applied mass forces, and we find the conditions that u must satisfy by equating to zero the 3×1 matrix $\text{div}_a T_a^*$ or, equivalently, since $T_a^* = T_a$ the 3×1 matrix $\text{div}_a T_a$ *Note* The axes of the space polar coordinate reference frame that is attached to the final position $(r + ku, \theta, \phi)$ of any particle of the medium have the same directions as the axes of the space polar coordinate reference frame that is attached to the initial position (r, θ, ϕ) of this particle In other words the rotation matrix R_1 that transforms the latter of these two reference frames into the

former is the identity matrix E_3. Thus the equation div$_a$ $(T_a{}^*R_1{}^*) = 0$, which expresses that the mass, or body, forces are zero, reduces to div$_a$ $(T_a{}^*) = 0$. In evaluating div$_a$ T_a we must use the expression for the divergence of a 3 × 3 matrix in space polar coordinates. This expression, and its derivation, are given in *Introduction to Applied Mathematics*, p. 112.[1] For a diagonal 3 × 3 matrix whose diagonal elements $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{\phi\phi}$ are functions of r alone, the divergence reduces to the 3 × 1 matrix

$$\left(\widehat{rr}' + \frac{2}{r}\widehat{rr} - \frac{1}{r}\widehat{\theta\theta} - \frac{1}{r}\widehat{\phi\phi}, \frac{\cot\theta}{r}(\widehat{\theta\theta} - \widehat{\phi\phi}), 0\right),$$

and since the second and third diagonal elements of T_a are equal it follows that u must satisfy the single condition furnished by the equation $\widehat{rr}' + \frac{2}{r}\widehat{rr} - \frac{1}{r}(\widehat{\theta\theta} + \widehat{\phi\phi}) = 0$.

We first obtain the solution furnished by the linear theory by neglecting powers of k above the first. T_a reduces to the diagonal matrix whose diagonal elements are

$$\widehat{rr} = k\left\{\lambda\left(u' + 2\frac{u}{r}\right) + 2\mu u'\right\},$$

$$\widehat{\theta\theta} = k\left\{\lambda\left(u' + 2\frac{u}{r}\right) + 2\mu\frac{u}{r}\right\} = \widehat{\phi\phi},$$

and so u must satisfy the differential equation

$$k(\lambda + 2\mu)\left(u'' + 2\frac{u'}{r} - 2\frac{u}{r^2}\right) = 0.$$

The general solution of this equation is $u = Ar + \frac{B}{r^2}$ where A and B are constants which must be determined by the facts that $T_a\,dS^a$ has the value $-p_i\,dS^x$ when $r = R_i$ and the value $-p_e\,dS^x$ when $r = R_e$. The 3 × 1 matrix dS^x is the product of the 3 × 1 matrix dS^a by $\left(1 + k\frac{u}{r}\right)^2$ (the second and third elements of each of these matrices being zero). Hence A and B are determined by setting $\widehat{rr} = -p_i\left(1 + k\frac{u}{r}\right)^2$ when $r = R_i$ and $\widehat{rr} = -p_e\left(1 + k\frac{u}{r}\right)^2$ when $r = R_e$. Thus p_i and p_e are

[1] By Francis D. Murnaghan, John Wiley & Sons, 1948.

of the same order of magnitude as the products of λ and μ by k and, to the degree of approximation afforded by the linear theory, we may replace $-p_i\left(1 + k\dfrac{u}{r}\right)^2$ and $-p_e\left(1 + k\dfrac{u}{r}\right)^2$ by $-p_i$ and $-p_e$, respectively When we substitute $u = Ar + \dfrac{B}{r^2}$ in the expression for $\widehat{rr}$ we obtain

$$\widehat{rr} = k\left\{(3\lambda + 2\mu)A - \frac{4\mu B}{r^3}\right\},$$

and so the two equations that serve to determine A and B are

$$(3\lambda + 2\mu)A - \frac{4\mu B}{R_i^3} = -\left(\frac{p_i}{k}\right)$$

$$(3\lambda + 2\mu)A - \frac{4\mu B}{R_e^3} = -\left(\frac{p_e}{k}\right)$$

Hence

$$A = \frac{p_iR_i^3 - p_eR_e^3}{k(3\lambda + 2\mu)(R_e^3 - R_i^3)}, \qquad B = \frac{(p_i - p_e)R_i^3R_e^3}{4k\mu(R_e^3 - R_i^3)}$$

We denote by u_1 this linear theory solution, thus the radial displacement ku_1 that is furnished by the linear theory is

$$ku_1 = \frac{p_iR_i^3 - p_eR_e^3}{(3\lambda + 2\mu)(R_e^3 - R_i^3)} + \frac{(p_i - p_e)R_i^3R_e^3}{4\mu(R_e^3 - R_i^3)}\frac{1}{r^2}$$

The solution furnished by the second order approximation is of the form $u = u_1 + kw$ where, in determining the equation that w must satisfy, we may replace u by u_1 in the coefficient of k^2 in T_a (for we neglect, in the second order approximation, powers of k above the second) T_a is the 3×3 diagonal matrix whose diagonal elements are furnished by the formulas

$$\widehat{rr} = k\left\{\lambda\left(u' + 2\frac{u}{r}\right) + 2\mu u'\right\} + k^2\left\{\lambda\left(\frac{3}{2}u'^2 + \frac{2uu'}{r} + \frac{u^2}{r^2}\right) + 3\mu u'^2 + l\left(u' + 2\frac{u}{r}\right)^2 + 2m\left(u'^2 - \frac{u^2}{r^2}\right) + n\frac{u^2}{r^2}\right\},$$

$$\widehat{\theta\theta} = k\left\{\lambda\left(u' + 2\frac{u}{r}\right) + 2\mu\frac{u}{r}\right\} + k^2\left\{\lambda\left(\frac{1}{2}u'^2 + \frac{uu'}{r} + 3\frac{u^2}{r^2}\right) + 3\mu\frac{u^2}{r^2} + l\left(u' + 2\frac{u}{r}\right)^2 + 2m\left(\frac{u^2}{r^2} - \frac{uu'}{r}\right) + n\frac{uu'}{r}\right\} = \widehat{\phi\phi}$$

where it is understood that, in the coefficients of k^2, u is to be replaced by u_1, whereas, in the coefficients of k, u is to be replaced by $u_1 + kw$. On calculating $\operatorname{div}_a T_a$ we find that the coefficient of k is zero (precisely because u_1 was determined by means of the linear theory), and on equating the coefficient of k^2 to zero we find that w must satisfy the equation

$$(\lambda + 2\mu)\left(w'' + 2\frac{w'}{r} - 2\frac{w}{r^2}\right) = 2(\lambda + 3\mu + 2m)\frac{\left(u_1' - \dfrac{u_1}{r}\right)^2}{r}$$
$$= \frac{18(\lambda + 3\mu + 2m)B^2}{r^7}.$$

Hence $w = Cr + \dfrac{D}{r^2} + \dfrac{(\lambda + 3\mu + 2m)B^2}{(\lambda + 2\mu)r^5}$ where C and D are two undetermined constants. The radial displacement furnished by the second-order approximation is

$$ku = ku_1 + k^2w = (kA + k^2C)r + \frac{(kB + k^2D)}{r^2}$$
$$+ \frac{(\lambda + 3\mu + 2m)k^2B^2}{(\lambda + 2\mu)r^5}.$$

The constants A and B have been determined, by the linear theory, so that the coefficient of k in $\widehat{rr} = -\left(\dfrac{p_i}{k}\right)$ when $r = R_i$, and $-\left(\dfrac{p_e}{k}\right)$ when $r = R_e$. We now determine C and D so that the coefficient of k^2 in $\widehat{rr} = -2p_i\dfrac{u_1}{rk}$, when $r = R_i$, and $= -2p_e\dfrac{u_1}{rk}$, when $r = R_e$. For this to be the case we must have

$$(3\lambda + 2\mu)C - \frac{4\mu D}{R_i^3} = \frac{-2p_i}{k}\left(A + \frac{B}{R_i^3}\right)$$
$$+ \frac{(3\lambda + 10\mu)(\lambda + 3\mu + 2m)}{\lambda + 2\mu}\frac{B^2}{R_i^6} - \lambda\left(\frac{9}{2}A^2 - \frac{6AB}{R_i^3} + \frac{3B^2}{R_i^6}\right)$$
$$- 3\mu\left(A - \frac{2B}{R_i^3}\right)^2 - 9lA^2 + 6mB\left(\frac{2A}{R_i^3} - \frac{B}{R_i^6}\right) - n\left(A + \frac{B}{R_i^3}\right)^2$$

and a similar equation obtained by replacing R_i by R_e and p_i by p_e. On replacing $\dfrac{-2p_i}{k}$ by $(6\lambda + 4\mu)A - \dfrac{8\mu B}{R_i^3}$ and collecting terms we obtain

$$(3\lambda + 2\mu)C - 4\mu \frac{D}{R_i^3} = \left(\frac{3}{2}\lambda + \mu - 9l - n\right)A^2$$
$$+ \frac{2AB}{R_i^3}(6\lambda + 4\mu + 6m - n) - \left(\frac{\mu(7\lambda + 10\mu - 8m)}{\lambda + 2\mu} + n\right)\frac{B^2}{R_i^6}$$

On replacing R_i by R_e we obtain a second equation connecting C and D, and on solving these two equations for C and D we obtain

$$C = \left(\frac{1}{2} - \frac{9l + n}{3\lambda + 2\mu}\right)A^2 + \left\{\frac{\mu(7\lambda + 10\mu - 8m)}{(\lambda + 2\mu)(3\lambda + 2\mu)} + \frac{n}{3\lambda + 2\mu}\right\}\frac{B^2}{R_i^3 R_e^3},$$

$$D = -\frac{1}{\mu}\left(3\lambda + 2\mu + 3m - \frac{1}{2}n\right)AB + \left\{\frac{(7\lambda + 10\mu - 8m)}{4(\lambda + 2\mu)} + \frac{n}{4\mu}\right\}\left(\frac{1}{R_i^3} + \frac{1}{R_e^3}\right)B^2$$

Special Cases

1 *The solid sphere* Here $R_i = 0$ and so (if we denote the applied pressure simply by p) $A = \dfrac{-p}{k(3\lambda + 2\mu)}$, $B = 0$, $C = \left(\dfrac{1}{2} - \dfrac{9l + n}{3\lambda + 2\mu}\right)\dfrac{p^2}{k^2(3\lambda + 2\mu)^2}$ $D = 0$ The radial displacement is

$$\left\{\frac{-p}{3\lambda + 2\mu} + \left(\frac{1}{2} - \frac{9l + n}{3\lambda + 2\mu}\right)\frac{p^2}{(3\lambda + 2\mu)^2}\right\} r$$

It is easy to verify that this expression agrees with the result furnished by the second order approximation in the treatment of the hydrostatic compression of an isotropic medium (carry out this verification) (*Hint* If the strain matrix is $-eE_3$ the coefficient of r above $= -e - \frac{1}{2}e^2$ since, if we denote this coefficient by δ, $(1 + \delta)^3 = (1 - 2e)^{3/2}$)

2 *The spherical cavity in an infinite solid* Here $p_2 = 0$, $R_2 = \infty$ and so (if we denote the pressure on the surface of the cavity simply by p and the initial radius of the cavity simply by R) $A = 0$, $B = \dfrac{p}{4k\mu}R^3$, $C = 0$, $D = \left\{\dfrac{7\lambda + 10\mu - 8m}{4(\lambda + 2\mu)} + \dfrac{n}{4\mu}\right\}\left(\dfrac{p}{4k\mu}\right)^2 R^3$ The radial displacement is $\left[\dfrac{p}{4\mu} + \left\{\dfrac{7\lambda + 10\mu - 8m}{4(\lambda + 2\mu)} + \dfrac{n}{4\mu}\right\}\left(\dfrac{p}{4\mu}\right)^2\right]\dfrac{R^3}{r^2}$

3 *The thin spherical shell* Denoting the thickness of the shell by ϵ, we have $R_e = R_i + \epsilon$ On keeping in the expressions for A and B only the lowest powers of $\dfrac{\epsilon}{R_i}$, we obtain

$$kA = \frac{(p_1 - p_2)}{3(3\lambda + 2\mu)} \frac{R_i}{\epsilon} \qquad \frac{kB}{R_i^3} = \frac{(p_1 - p_2)}{12\mu} \frac{R_i}{\epsilon} = \beta,$$

say; therefore, $kA = \dfrac{4\mu}{3\lambda + 2\mu}\beta$. It follows from the expressions for C and D that

$$k^2C = \left\{\left(\frac{1}{2} - \frac{9l + n}{3\lambda + 2\mu}\right)\frac{16\mu^2}{(3\lambda + 2\mu)^2} + \frac{\mu(7\lambda + 10\mu - 8m)}{(\lambda + 2\mu)(3\lambda + 2\mu)} + \frac{n}{3\lambda + 2\mu}\right\}\beta^2.$$

$$\frac{k^2D}{R_i^3} = \left\{\frac{-4}{3\lambda + 2\mu}\left(3\lambda + 2\mu + 3m - \frac{1}{2}n\right) + \frac{(7\lambda + 10\mu - 8m)}{2(\lambda + 2\mu)} + \frac{n}{2\mu}\right\}\beta^2.$$

The radial displacement is found by substituting these values in the expression

$$\left\{kA + k^2C + \frac{kB + k^2D}{r^3} + \frac{(\lambda + 3\mu + 2m)k^2B^2}{(\lambda + 2\mu)r^6}\right\} r.$$

whose magnitude is a function of the initial distance r of the particle from the axis of the tube. If we denote this displacement by ku, the plane polar coordinates (relative to a reference frame whose origin is on the axis of the tube) of the final position of a particle whose initial position had the plane polar coordinates (r, θ) are $(r + ku, \theta)$. Here k is a constant which is such that in the linear theory we neglect powers of k above the first whereas in the second order approximation we neglect powers of k above the second. The plane polar coordinates of the vector **da** are $(dr, r\,d\theta)$, and the plane polar coordinates of the vector **dx** are $(dr + k\,du, (r + ku)\,d\theta)$. Thus da is the 2×1 matrix $\begin{pmatrix} dr \\ r\,d\theta \end{pmatrix}$ whereas dx is the 3×1 matrix $\begin{pmatrix} dr + k\,du \\ (r + ku)\,d\theta \end{pmatrix}$ and the 2×2 matrix J that transforms by means of the formula $dx = J\,da$, da into dx is the diagonal matrix

$$J = \begin{pmatrix} 1 + ku & 0 \\ 0 & 1 + k\dfrac{u}{r} \end{pmatrix}$$

where the prime denotes differentiation with respect to r. It follows that the plane polar coordinates of the (plane) strain matrix η are furnished by the elements of the 2×2 diagonal matrix

$$\eta = k\begin{pmatrix} u' & 0 \\ 0 & \dfrac{u}{r} \end{pmatrix} + \frac{1}{2}k^2\begin{pmatrix} u'^2 & 0 \\ 0 & \dfrac{u^2}{r^2} \end{pmatrix}$$

Neglecting powers of k above the second we have

$$I_1 = k\left(u' + \frac{u}{r}\right) + \frac{1}{2}k^2\left(u'^2 + \frac{u^2}{r^2}\right)$$

$$I_2 = k^2\frac{uu'}{r}$$

Since the medium is by hypothesis isotropic and free from initial stress,

$$\frac{\partial\phi}{\partial\eta} = \lambda I_1 E_3 + 2\mu\eta + (lI_1^2 - 2mI_2)E_3 + 2mI_1\eta$$

Note. Since we are dealing with a two-dimensional problem so that the strain matrix η is two-dimensional rather than three dimensional, the invariant I_3 does not appear and we have only the two third order elastic constants l and m rather than the three third order elastic constants l, m, and n.

$\frac{\partial \phi}{\partial \eta}$ and $T_a = J \frac{\partial \phi}{\partial \eta}$ are diagonal. We assume that our tube is free from applied mass forces and so we find the conditions that u must satisfy by setting $\operatorname{div}_a T_a = 0$ (why?). For a diagonal 2×2 matrix whose diagonal elements $\widehat{rr}$, $\widehat{\theta\theta}$ are functions of r alone, the divergence (in plane polar coordinates) reduces to the 2×1 matrix

$$\left(\widehat{rr}' + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}), 0\right)$$

(see *Introduction to Applied Mathematics*, p. 98)[1] and therefore u must satisfy the single condition furnished by the equation $\widehat{rr}' + \frac{1}{r}(\widehat{rr} - \widehat{\theta\theta}) = 0$.

We first obtain the linear theory solution by neglecting powers of k above the first. T_a reduces, then, to the 2×2 diagonal matrix whose diagonal elements are

$$\widehat{rr} = k\left\{(\lambda + 2\mu)u' + \lambda \frac{u}{r}\right\},$$

$$\widehat{\theta\theta} = k\left\{\lambda u' + (\lambda + 2\mu)\frac{u}{r}\right\}$$

and so u must satisfy the differential equation

$$ru'' + u' - \frac{u}{r} = 0.$$

The general solution of this equation is $u = Ar + \frac{B}{r}$ where A and B are constants which must be determined by the facts that $T_a\, dS^a$ has the value $-p_i\, dS^x$ when $r = R_i$ and the value $-p_e\, dS^x$ when $r = R_e$. The 2×1 matrix dS^x is the product of the 2×1 matrix dS^a by $1 + k\frac{u}{r}$; thus A and B are determined by setting $\widehat{rr} = -p_i\left(1 + k\frac{u}{r}\right)$ when $r = R_i$ and $\widehat{rr} = -p_e\left(1 + k\frac{u}{r}\right)$ when $r = R_e$. It follows that p_i and p_e are of the same order of magnitude as the products of λ and μ by k, and so (to the degree of approximation afforded by the linear theory) we may equate $\widehat{rr}$ to $-p_i$ when $r = R_i$ and to $-p_e$ when $r = R_e$. When we substitute $u = Ar + \frac{B}{r}$ in the expression for

[1] By Francis D. Murnaghan, John Wiley & Sons, 1948.

$\dfrac{p^2}{4k^2(\lambda+\mu)^2}$, $D = 0$ The radial displacement is

$$\frac{-pr}{2(\lambda+\mu)}\left\{1+\left(2+\frac{2l+m}{\lambda+\mu}\right)\frac{p}{2(\lambda+\mu)}\right\}$$

2 *The thin-walled circular tube* Denoting the thickness of the walls of the tube by ϵ we have $R_e = R_i + \epsilon$ On keeping in the expressions for A and B only the lowest powers of $\dfrac{\epsilon}{R_i}$, we obtain

$$kA = \frac{p_i - p_e}{4(\lambda+\mu)}\frac{R_i}{\epsilon}, \qquad \frac{kB}{R_i^2} = \frac{(p_i - p_e)}{4\mu}\frac{R_i}{\epsilon} = \beta,$$

say, and therefore $kA = \dfrac{\mu}{\lambda+\mu}\beta$ C and D are furnished by the formulas

$$k^2C = -\left\{\left(2+\frac{2l+m}{\lambda+\mu}\right)\left(\frac{\mu}{\lambda+\mu}\right)+\frac{\mu+2m}{\lambda+2\mu}\right\}\frac{\mu}{\lambda+\mu}\beta^2,$$

$$k^2D = -\left\{\frac{3\lambda+6\mu+4m}{\lambda+\mu}+\frac{2(\mu+2m)}{\lambda+2\mu}\right\}\beta^2$$

The radial displacement of any point on the inner wall of the thin tube is

$$(ku)_{r=R_i} = \left[\frac{2\mu\beta}{\lambda+\mu} - \left\{\left(2+\frac{2l+m}{\lambda+\mu}\right)\left(\frac{\mu}{\lambda+\mu}\right)^2+\frac{\mu(\mu+2m)}{(\lambda+\mu)(\lambda+2\mu)}\right.\right.$$
$$\left.\left.+\frac{3\lambda+6\mu+4m}{\lambda+\mu}+\frac{\mu+6m-\lambda}{2(\lambda+2\mu)}\right\}\beta^2\right]R_i$$

3. The torsion of a circular cylinder

We choose the axis of our cylinder as the axis of a system of cylindrical coordinates (r, θ, z), and we denote by $r + ku$, $\theta + kz$, $z + kw$ the final coordinates of the particle of the cylinder whose initial coordinates were (r, θ, z) Here k is a constant parameter which is such that its square and higher powers may be neglected in the linear theory whereas its cube and higher powers may be neglected in the second-order approximation We assume that u is a function of r alone, that w is a function of z alone, and that the medium is isotropic and free from initial stress, and we try to so determine the unknown functions $u = u(r)$, $w = w(z)$ that

(1) The deformed cylinder is in equilibrium under the action of surface forces alone, the mass or body forces being zero.

(2) There are no applied forces on the sides (or curved portion of the bounding surface) of the cylinder, and the forces applied to each end of the cylinder reduce to a couple about the axis of the cylinder.

The cylindrical coordinates of the vector **da** are $(dr, r\,d\theta, dz)$ whereas the cylindrical coordinates of the vector **dx** are $\{(1 + ku')\,dr, \left(1 + k\frac{u}{r}\right)(r\,d\theta + kr\,dz), (1 + kw')\,dz\}$ where u' denotes the derivative of u with respect to r and w' denotes the derivative of w with respect to z. *Note.* The cylindrical coordinates of **dx** have reference to a frame obtained by rotating the reference frame in which the cylindrical coordinates of **da** are $(dr, r\,d\theta, dz)$ through an angle kz around the z-axis. The 3×3 matrix that transforms, by means of the formula $dx = J\,da$, the 3×1 matrix

$$da = \begin{pmatrix} dr \\ r\,d\theta \\ dz \end{pmatrix}$$

into the 3×1 matrix

$$dx = \begin{pmatrix} (1 + ku')\,dr \\ \left(1 + k\dfrac{u}{r}\right) r\,d\theta + k(r + ku)\,dz \\ (1 + kw')\,dz \end{pmatrix}$$

is

$$J = \begin{pmatrix} 1 + ku' & 0 & 0 \\ 0 & 1 + k\dfrac{u}{r} & k(r + ku) \\ 0 & 0 & 1 + kw' \end{pmatrix}.$$

Hence the cylindrical coordinates of the strain matrix η are given by the formula

$$\eta = k\begin{pmatrix} u' & 0 & 0 \\ 0 & \dfrac{u}{r} & \dfrac{1}{2}r \\ 0 & \dfrac{1}{2}r & w' \end{pmatrix} + k^2\begin{pmatrix} \dfrac{1}{2}(u')^2 & 0 & 0 \\ 0 & \dfrac{1}{2}\dfrac{u^2}{r^2} & u \\ 0 & u & \dfrac{1}{2}(w')^2 + r^2 \end{pmatrix}$$

(powers of k above the second being neglected). It follows that

$$I_1 = k\left(u' + \frac{u}{r} + w'\right) + k^2\left\{\frac{1}{2}(u')^2 + \frac{1}{2}\frac{u^2}{r^2} + \frac{1}{2}(w')^2 + r^2\right\}.$$

$$I_2 = k^2 \left\{\frac{uu'}{r} + \left(u' + \frac{u}{r}\right) w' - \frac{1}{4} r^2\right\},$$

$$\text{co } \eta = k^2 \begin{pmatrix} \dfrac{uw'}{r} - \dfrac{1}{4} r^2 & 0 & 0 \\ 0 & u'w' & -\dfrac{1}{2} ru' \\ 0 & -\dfrac{1}{2} ru' & \dfrac{uu'}{r} \end{pmatrix}$$

We first determine the linear theory solution Here

$$\frac{\partial \phi}{\partial \eta} = k \begin{pmatrix} \lambda\left(u' + \dfrac{u}{r} + w'\right) + 2\mu u' & 0 & 0 \\ 0 & \lambda\left(u' + \dfrac{u}{r} + w'\right) + 2\mu \dfrac{u}{r} & \mu r \\ 0 & \mu r & \lambda\left(u' + \dfrac{u}{r} + w'\right) + 2\mu w' \end{pmatrix}$$

and $T_a = J \dfrac{\partial \phi}{\partial \eta}$ is the same as this Since we are using orthogonal curvilinear coordinates the matrix whose divergence must be equated to zero (to express the fact that the mass or body, forces are zero) is $T_a{}^*R_1{}^*$ where R_1 is the rotation matrix that transforms the cylindrical coordinate reference frame at (r, θ, z) into the cylindrical coordinate reference frame at $(r + ku, \theta + kz, z + kw)$

$$R_1 = \begin{pmatrix} \cos kz & -\sin kz & 0 \\ \sin kz & \cos kz & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_3 + k \begin{pmatrix} 0 & -z & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

Since we are neglecting (when obtaining the linear theory solution) powers of k above the first the matrix whose divergence must be set equal to zero is

$$k \begin{pmatrix} \lambda\left(u' + \dfrac{u}{r} + w'\right) + 2\mu u' & 0 & 0 \\ 0 & \lambda\left(u' + \dfrac{u}{r} + w'\right) + 2\mu \dfrac{u}{r} & \mu r \\ 0 & \mu r & \lambda\left(u' + \dfrac{u}{r} + w'\right) + 2\mu w' \end{pmatrix}$$

Using the formula for the divergence of a 3×3 matrix in cylindrical

coordinates (see *Introduction to Applied Mathematics*, p. 112),[1] we obtain the following equations:

$$u'' + \frac{u'}{r} - \frac{u}{r^2} = 0, \qquad w'' = 0.$$

Hence u is a linear combination of r and $\dfrac{1}{r}$ and, since u must be bounded at $r = 0$, it follows that u is a multiple of r. Also w' is a constant and so we may take, without loss of generality (why?), w to be a multiple of z:

$$u = Ar, \qquad w = Bz.$$

The fact that there is no applied force on the sides of the cylinder yields $2(\lambda + \mu)A + \lambda B = 0$. The coordinate in the direction of the axis of the cylinder of the force, per unit initial area, on the end $z = l$ of the cylinder is constant and equals $2\lambda A + (\lambda + 2\mu)B$. In order that the force on the end $z = l$ of the cylinder should reduce to a couple we must have, then, $2\lambda A + (\lambda + 2\mu)B = 0$. The two equations $2(\lambda + \mu)A + \lambda B = 0$, $2\lambda A + (\lambda + 2\mu)B = 0$ yield $A = 0$, $B = 0$. Thus the solution furnished by the linear theory is $u_1 = 0$, $w_1 = 0$; thus any cross section z = constant of the cylinder is merely rotated through an angle kz. The force, per unit initial area, applied to the end $z = l$ of the cylinder is perpendicular to the radius to the point of application and is of magnitude $k\mu r$. Hence the system of forces applied to the end $z = l$ of the cylinder is equivalent to a couple or torque around the axis of the cylinder of amount $2\pi \int_0^R k\mu r^2\, dr = \dfrac{k\pi\mu R^4}{2}$. If α is the angle through which the end $z = l$ of the cylinder is rotated we have $\alpha = kl$ and thus we obtain the formula

$$\text{Torque} = \frac{\alpha\pi\mu R^4}{2l}$$

which serves to determine the *modulus of rigidity* μ.

In order to obtain the solution furnished by the second-order approximation, we set, since u_1 and w_1 are zero, $u = krv$, and $w = ks$ where v and s are undetermined functions of r and z, respectively;

$$v = v(r); \qquad s = s(z).$$

Neglecting powers of k above the second, we obtain

[1] *Ibid.*

$$\eta = k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}r \\ 0 & \frac{1}{2}r & 0 \end{pmatrix} + k^2\begin{pmatrix} v' & 0 & 0 \\ 0 & \dfrac{v}{r} & 0 \\ 0 & 0 & s' + \frac{1}{2}r^2 \end{pmatrix}$$

where v' denotes the derivative of v with respect to r and s' denotes the derivative of s with respect to z. Hence

$$I_1 = k^2\left(v' + \frac{v}{r} + s' + \frac{1}{2}r^2\right)$$

$$I_2 = -\tfrac{1}{4}k^2r^2$$

$$\text{co}\ \eta = -\tfrac{1}{4}k^2\begin{pmatrix} r^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so

$$\frac{\partial\phi}{\partial\eta} = k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu r \\ 0 & \mu r & 0 \end{pmatrix}$$

$$+ k^2\begin{bmatrix} \begin{matrix}\lambda\left(v' + \frac{v}{r} + s'\right) + 2\mu v' \\ + \left(\frac{1}{2}\lambda + \frac{1}{2}m - \frac{1}{4}n\right)r^2\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}\lambda\left(v' + \frac{v}{r} + s'\right) + 2\mu\frac{v}{r} \\ + \left(\frac{1}{2}\lambda + \frac{1}{2}m\right)r^2\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}\lambda\left(v' + \frac{v}{r} + s'\right) + 2\mu s' \\ + \left(\frac{1}{2}\lambda + \mu + \frac{1}{2}m\right)r^2\end{matrix} \end{bmatrix}$$

The matrix whose divergence must be equated to zero (to express the fact that the mass or body forces are zero) is $\dfrac{\partial\phi}{\partial\eta}J^*R_1^*$ where

$$R_1 = E_3 + k\begin{pmatrix} 0 & -z & 0 \\ z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \qquad , J = E_3 + k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} +$$

Hence

$$R_1J = E_3 + k\begin{pmatrix}0 & -z & 0\\ z & 0 & r\\ 0 & 0 & 0\end{pmatrix} + \cdots,$$

$$J^*R_1^* = E_3 + k\begin{pmatrix}0 & z & 0\\ -z & 0 & 0\\ 0 & r & 0\end{pmatrix} + \cdots \text{ and so}$$

$$\frac{\partial\phi}{\partial\eta}J^*R_1^* = k\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & r\\ 0 & r & 0\end{pmatrix}$$

$$+ k^2\begin{bmatrix}\lambda\left(v' + \frac{v}{r} + s'\right) + 2\mu v' + \left(\frac{1}{2}\lambda + \frac{1}{2}m - \frac{1}{4}n\right)r^2 & 0 & 0\\ 0 & \lambda\left(v' + \frac{v}{r} + s'\right) + 2\mu\frac{v}{r} + \left(\frac{1}{2}\lambda + \mu + \frac{1}{2}m\right)r^2 & 0\\ -\mu rz & 0 & \lambda\left(v' + \frac{v}{r} + s'\right) + 2\mu s' + \left(\frac{1}{2}\lambda + \mu + \frac{1}{2}m\right)r^2\end{bmatrix}.$$

On equating the divergence of this matrix to zero, we obtain the following equations:

$$(\lambda + 2\mu)\left(v'' + \frac{v'}{r} - \frac{v}{r^2}\right) = \left(2\mu - \lambda - m + \frac{3}{4}n\right)r, \qquad s'' = 0.$$

Since v is bounded at $r = 0$, we have

$$v = Cr + \frac{(2\mu - \lambda - m + \frac{3}{4}n)r^3}{8(\lambda + 2\mu)}$$

where C is an undetermined constant. s is a linear function of z, and we may set (why?)

$$s = Dz$$

where D is an undetermined constant. Since $T_e = J\dfrac{\partial\phi}{\partial\eta}$ is given by the formula

$$T_a = k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu r \\ 0 & \mu r & 0 \end{pmatrix} +$$

$$k^2 \begin{bmatrix} \begin{matrix}\lambda\left(v' + \dfrac{v}{r} + s'\right) + 2\mu v' \\ + \left(\dfrac{1}{2}\lambda + \dfrac{1}{2}m - \dfrac{1}{4}n\right) r^2\end{matrix} & 0 & 0 \\ 0 & \begin{matrix}\lambda\left(v' + \dfrac{v}{r} + s'\right) + 2\mu \dfrac{v}{r} \\ + \left(\dfrac{1}{2}\lambda + \mu + \dfrac{1}{2}m\right) r^2\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}\lambda\left(v' + \dfrac{v}{r} + s'\right) + 2\mu s' \\ + \left(\dfrac{1}{2}\lambda + \mu + \dfrac{1}{2}m\right) r^2\end{matrix} \end{bmatrix},$$

the fact that there is no applied force on the sides of the cylinder yields

$$2(\lambda + \mu)C + \lambda D + \frac{R^2}{16(\lambda + 2\mu)}\{20\lambda\mu + 24\mu^2 + 4\mu m + (2\lambda + \mu)n\} = 0$$

where R is the initial radius of the cylinder The axial coordinate of the force per unit initial area applied to the end $z = l$ of the cylinder is the product of k^2 by $2\lambda C + (\lambda + 2\mu)D + \dfrac{24\lambda\mu + 16\mu^2 + 8\mu m + 3\lambda n}{8(\lambda + 2\mu)} r^2$, and we choose the constants C and D so that the integral of this over the end of the cylinder is zero Thus we obtain the relation

$$2\lambda C + (\lambda + 2\mu)D + \frac{24\lambda\mu + 16\mu^2 + 8\mu m + 3\lambda n}{16(\lambda + 2\mu)} R^2 = 0$$

When C and D satisfy the two linear equations just determined, the force applied to the end $z = l$ is a couple about the axis of the cylinder of amount $\dfrac{\pi\mu k R^4}{2}$ (prove this) Under this moment (or *torque*) not only does the cylinder rotate (as predicted by the linear theory) but also its radius increases by the amount

$$k^2 R\left\{C + \frac{(2\mu - \lambda - m + \frac{3}{4}n)}{8(\lambda + 2\mu)} R^2\right\}$$

and its length increases by the amount k^2Dl. Since C and D are multiples of R^2 it follows that the increase of the radius of the cylinder is a multiple (possibly negative) of k^2R^3 whereas the increase of the length of the cylinder is a multiple (possibly negative) of k^2R^2. Accurate measurements of the changes of the radii and lengths of cylinders (of reasonably large radius) would serve to determine the third-order elastic constants m and n (it being supposed that the second-order elastic constants λ and μ are already known).

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